Calculus II – 02
Pyramid Exam 1 – Take-Home Portion
Each part of each problem is equally weighted. Work all problems on separate pages.
Problems #1,2,3,7 & 8 refer to one of the following sequences:

\[ a_n = \frac{(-1)^{n+1} n}{n^2 + 1} \quad b_n = \frac{n^2 - n + 1}{n^3 + 3n + 2} \quad f_n = \frac{n^3 + n}{2^n} \]

1.
   a. Write the first six terms of the sequence \( a_n \) defined above.
   \( 1/2, -2/5, 3/10,-4/17, 5/26, -6/37 \)
   b. Write an expression for a sequence \( r_n \) so that \( r_1 = -1/2, r_2 = 2/7, \)
      \( r_3 = -3/12, r_4 = 4/17, \) and \( r_5 = -5/22. \)
      The alternating, starting with a negative number, is accomplished by a factor \((-1)^n\). The numerators (their absolute values, anyway) agree with the subscripts, so they are attained by the factor \( n \), while the denominators increase by 5 starting with the number 2, which can be accomplished by \( 2+5(n-1) \) or \( 5n-3. \) Putting them all together, \( r_n = \frac{(-1)^n n}{5n-3} \) works.
   c. Write an expression for a sequence \( t_n \) so that \( t_1 = 1/2, t_2 = -4/8, \)
      \( t_3 = 7/32, t_4 = -10/128, \) and \( t_5 = 13/512. \)
      The alternating, starting with a positive number, is accomplished by a factor \((-1)^{n+1}\). The numerators (their absolute values, anyway) increase by 3 starting with the number 1, which can be accomplished by \( 1+3(n-1) \) or \( 3n-2. \), while the denominators are all powers of 2, with exponents 1, 3, 5, 7, 9, which can be accomplished by \( 2n-1. \) Putting them all together, \( t_n = \frac{(-1)^{n+1}(3n-2)}{2^{2n-1}} \) works.

2.
   a. Determine whether the limit of the nth term of the series \( \sum_{n=1}^{\infty} a_n \),
      where \( a_n \) is defined above, exists. If it does exist, what is it?
      \[ a_n = \frac{(-1)^{n+1} n}{n^2 + 1} = \frac{(-1)^{n+1} n / n}{(n^2 + 1) / n} = \frac{(-1)^{n+1}}{n + 1 / n} \]
      so \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left\{ \frac{(-1)^{n+1}}{n + 1 / n} \right\} = 0, \) since the denominator clearly approaches infinity and the numerator is bounded.
b. Change the numerator (but not the denominator) of the sequence $a_n$ defined above so that the limit of the $n$th term of the series $\sum_{n=1}^{\infty} a_n$ is the number 3.

To get the limit to be greater than 0 but not infinity, we first adjust the numerator so that it is a polynomial the same degree as that of the denominator and delete the alternating factor: Take

$$a'_n = \frac{n^2}{n^2 + 1} = \frac{n^2 / n^2}{(n^2 + 1)/n^2} = \frac{1}{(1+1/n^2)}$$

which clearly approaches 1 as $n$ approaches infinity. To get the limit 3, multiply the numerator by 3 to get the sequence meeting the requirement: $a''_n = \frac{3n^2}{n^2 + 1}$.

3.

a. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n$ is defined above originally, not as you modified it in part b above. Determine whether the limit of the sequence of partial sums of this series exists.

To say that the limit of the sequence of partial sums of this series exists just means that the series converges. The series is alternating, every term is non-zero and the limit of the $n$th term is zero, so if we can show that the absolute values of the terms form a decreasing sequence, then the Alternating Series Test, Theorem 9.8, will show that the series converges.

We need to show: $\frac{(n+1)}{(n+1)^2 + 1} < \frac{n}{n^2 + 1}$ for $n>0$, which is equivalent to showing that: $(n+1)(n^2+1) < n((n+1)^2 + 1]$, which, in turn, is equivalent to:

$$n^3 + n^2 + n + 1 < n^3 + 2n^2 + n + 1$$

which is clearly true!

b. Again considering the series $\sum_{n=1}^{\infty} a_n$, determine whether it converges absolutely.

The series of absolute values of $a_n$ is similar to the Harmonic series, so we’ll compare it to the Harmonic series. The Comparison Test works if we compare term-by-term, but starting with the second term of the usual Harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ We need to show that $\frac{n}{n^2 + 1} > \frac{1}{n+1}$ (the $n$th term $a_n >$ the $n+1$st term of the Harmonic series). But the above inequality is
equivalent to \( n(n+1) > n^2 + 1 \), or \( n^2 + n > n^2 + 1 \), which is clearly true when \( n > 1 \).

Since the Harmonic series diverges, so does the Harmonic series starting with 1/2, instead of 1, and so by the Comparison Test, \( \sum_{n=1}^{\infty} a_n \) does not converge absolutely; i.e., \( \sum_{n=1}^{\infty} |a_n| \) diverges. Since the series \( \sum_{n=1}^{\infty} a_n \) converges but does not converge absolutely, it is said to converge conditionally.

(Alternatively, we could use the Limit Comparison Test on \( \sum_{n=1}^{\infty} |a_n| \), again comparing it to the Harmonic. The ratio of their nth terms is

\[
\frac{n}{n^2 + 1} = \frac{n^2}{1 + \left(\frac{1}{n^2}\right)}
\]

which diverges.)

4.

a. Construct, if possible, a convergent infinite geometric series whose 1st term is 3 and whose sum is 4. (If not possible, state why.)

A convergent geometric series has the form \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \), where \( |r| < 1 \). If the first term is to be 3 and the sum 4, then \( \frac{3}{1-r} = 4 \), and hence \( r = \frac{1}{4} \).

Since \( 0 < \frac{1}{4} < 1 \), the desired geometric series is \( \sum_{n=0}^{\infty} 3 \left(\frac{1}{4}\right)^n \).

b. Is it possible to construct a convergent infinite geometric series whose sum is 4 if its 1st term is 5?

For this one, using the same notation as in part a, we need \( \frac{5}{1-r} = 4 \), which is true for \( r = -\frac{1}{4} \). The series \( \sum_{n=0}^{\infty} 5 \left(-\frac{1}{4}\right)^n \) meets the requirements here.

c. Determine all the possible sums to which an infinite geometric series can converge if its 1st term is 5?
Method 1: \( S \) is a sum of an infinite geometric series whose 1\(^{st} \) term is 5 if and only if \( \frac{5}{1-r} = S \) for some number \( r \) for which \( |r|<1 \). The condition \( |r|<1 \) gives the following implications: \(-1 < r < 1 \iff -r > -1 \iff 2 > 1-r > 0 \Rightarrow S > 0 \)

The condition \( \frac{5}{1-r} = S \) is equivalent to \( \frac{5}{S} = 1-r \) and also \( r = 1 - \frac{5}{S} \), where \( |r| < 1 \). Now we get the condition on \( S \) that \(-1 < 1 - \frac{5}{S} < 1 \iff 2 < -\frac{5}{S} < 0 \iff \frac{5}{S} > 0 \) and finally, multiplying each side of the inequalities by \( S \) and dividing by 2 (and since \( S > 0 \)), \( S > \frac{5}{2} \).

Method 2: Since the sum of a convergent infinite geometric series whose 1\(^{st} \) term is 5 satisfies: \( \frac{5}{1-r} = S \), where \( S \) is the sum and \( |r|<1 \), consider the function \( f \) defined by \( f(x) = \frac{5}{1-x} \) for \(-1 \leq x < 1 \). By the Chain Rule, \( f'(x) = \frac{5}{(1-x)^2} \), which is positive for all \( x \) in the interval \([-1 , 1)\). Thus \( f \) is an increasing function on \([-1 , 1)\) and takes on its minimum at \( x = -1 \); this minimum is \( f(-1) = \frac{5}{2} \). Since \( \lim_{x \to 1} \frac{5}{1-x} = \infty \) and \( f \) is continuous on \([-1 , 1)\), \( S \) can be any real number greater than \( \frac{5}{2} \). (Note that the ratio must be strictly greater than \(-1\).)

5. If \( \sum_{n=1}^{\infty} c_n = 5 \) and \( \sum_{n=1}^{\infty} d_n = -10 \), what can you say about:

\[ \sum_{n=1}^{\infty} (7c_n - 2d_n) \]

Theorem 9.2.1 implies that \( \sum_{n=1}^{\infty} (7c_n - 2d_n) = 7(5) - 2(-10) = 55 \).

6. a. Show that \( \sum_{n=1}^{\infty} \frac{3^n}{2^n} \) diverges by giving an argument based on a part of Theorem 9.2, page 451.

The limit of the \( n \)th term of the sequence, as \( n \) approaches infinity,

\[ \lim_{n \to \infty} \frac{3^n}{2^n} = \lim_{n \to \infty} \left( \frac{3}{2} \right)^n = \infty \] according to the bulleted remark in the middle of page 440, so by Theorem 9.2, part 3, page 451 the series diverges.
b. Determine whether \( \sum_{n=1}^{\infty} \frac{2^n}{3^n} \) converges or diverges, and state an argument to convince someone of your assertion.

The series is a geometric series whose first term is \( 2/3 \) and whose common ratio is also \( 2/3 \). By the blue boxed statement and remarks following on page 446, this series converges to \( \frac{2}{1 - \frac{2}{3}} = 2 \).
7. a. Use the Comparison Test to show that the series \( \sum_{n=1}^{\infty} b_n \), where \( b_n \) is defined above, converges.

Compare \( \sum_{n=1}^{\infty} b_n \), whose nth term is \( \frac{n^2 - n + 1}{n^4 + 3n + 2} \) to the convergent series (called the 2-series) whose nth term is \( \frac{1}{n^2} \). To show that \( \sum_{n=1}^{\infty} b_n \) converges, it suffices to show that \( \frac{n^2 - n + 1}{n^4 + 3n + 2} < \frac{1}{n^2} \) or, equivalently, that \( (n^4 - n^3 + n^2) < n^4 + 3n + 2 \), which is equivalent to \( n^2 - 3n - 2 < n^3 \) or \( 1 - \frac{3}{n} - \frac{3}{n^2} < n \), which is clearly true for \( n > 1 \).

b. Determine whether the series \( \sum_{n=1}^{\infty} 7b_n \), where \( b_n \) is defined above, converges or diverges, and state an argument to convince someone of your assertion.

Since, according to part a, \( \sum_{n=1}^{\infty} b_n \) converges, the series \( \sum_{n=1}^{\infty} 7b_n \), also converges by Theorem 9.2, part 1, the second bulleted statement.

8. Use the Ratio Test to determine whether the series \( \sum_{n=1}^{\infty} f_n \), where \( f_n \) is defined above, converges.

We consider the ratio of the \( n+1^{st} \) term of the series to the nth term of the series:

\[
\frac{f_{n+1}}{f_n} = \frac{(n+1)^3 + n + 1}{2^{n+1}} \cdot \frac{n^3 + n}{2^n} = \left[ \frac{(n+1)^3 + n + 1}{2^{n+1}} \right] \cdot \left[ \frac{2^n}{n^3 + n} \right] = \left[ \frac{1}{2} \right] \left[ \frac{(n^3 + 3n^2 + 3n + 1 + n + 1)}{n^3 + n} \right]
\]

which simplifies to:
\[
\left[ \frac{1}{2} \right] \left[ (1 + \frac{3}{n} + \frac{4}{n^2} + \frac{2}{n^3}) \right].
\]

Since all the terms involving powers of \( n \) in the denominator approach 0 as \( n \) approaches infinity, the limit of this ratio is 1/2, and by the Ratio test, the series converges.