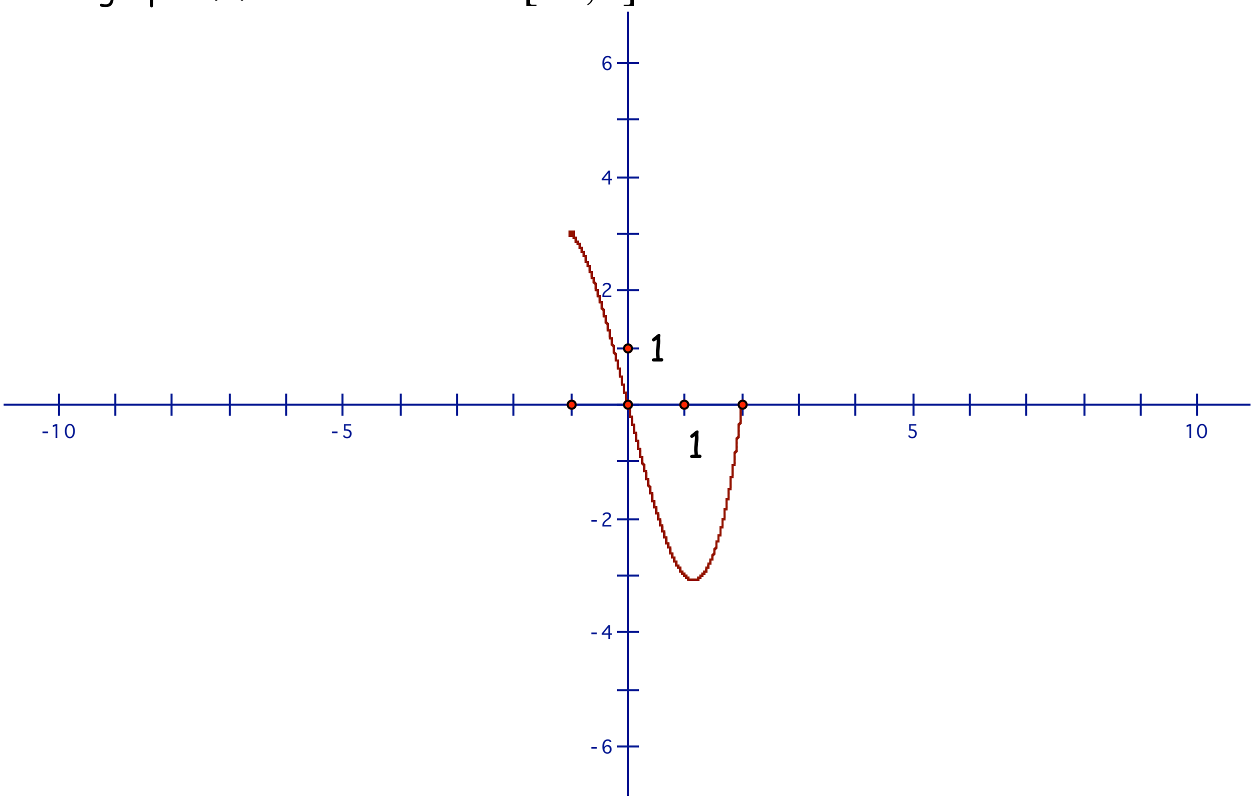
Integral Example
Math 101 – 01

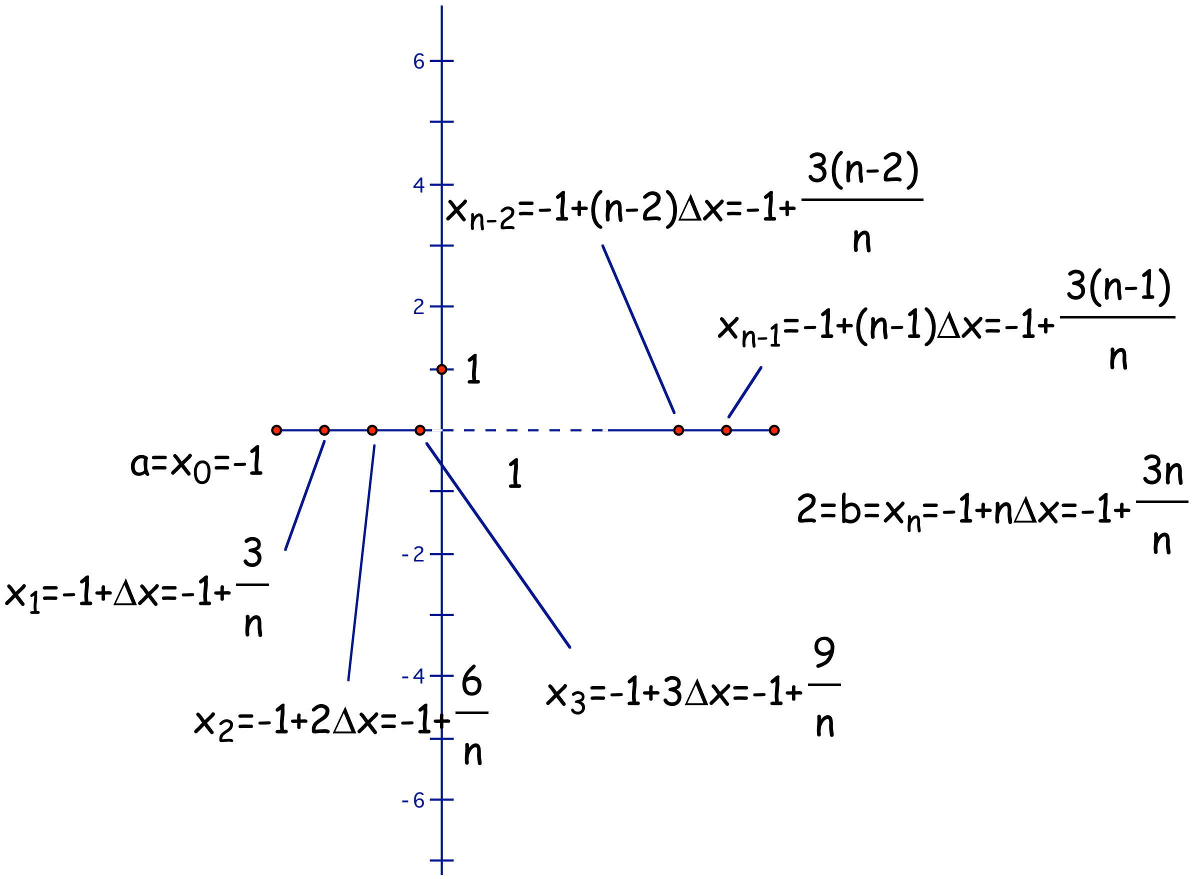
The example below illustrates the use of Theorem 4, p 264, and how to construct a specific form of Riemann Sum whose limit is the definite integral.

Here's the integral to evaluate: The integral of the function f, defined by $f(x)=x^3-4x$, over the interval $[-1,2]$.

1. The first thing is to get a rough picture of the function and interval: The graph of f over the interval $[-1,2]$ looks like:



2. The integral is a limit of Riemann sums, which depend on a partition of the interval over which the integral is taken. Guided by the requirements of Theorem 4, the interval $[-1,2]$ is divided into n equal subintervals, each of width $\frac{2-(-1)}{n}=\frac{3}{n}$. Using the notation of Theorem 4, $\Delta x=\frac{3}{n}$:



3. Again, guided by the requirements of Theorem 4, sample points: $x_1^*, x_2^*, x_3^*, \ldots, x_{n-2}^*, x_{n-1}^*, x_n^*$ are chosen to be right endpoints of each of the subintervals: $x_1^* = x_1 = -1 + \frac{3}{n}$, $x_2^* = x_2 = -1 + \frac{6}{n}$, $x_3^* = x_3 = -1 + \frac{9}{n}$, … ,

$x_i^* = x_i = -1 + \frac{3i}{n}$, … , $x_n^* = x_n = -1 + \frac{3n}{n} = 2$

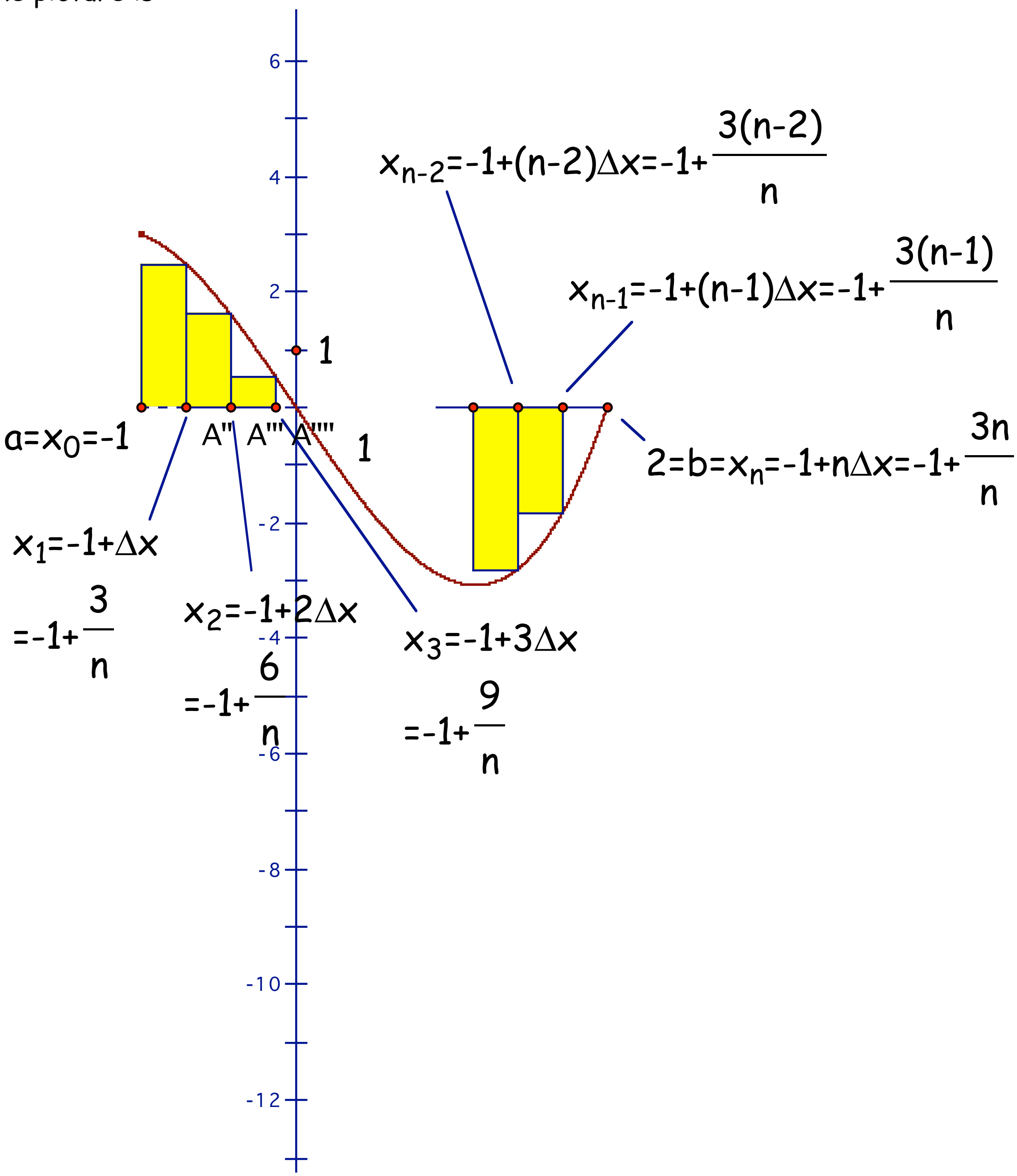
4. Now the Riemann Sum can be formed for this regular partition into n subintervals with sample points the right endpoints of the subintervals: For now in this example, I'll avoid using the sigma notation.)

$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \ldots + f(x_i^*)\Delta x + \ldots + f(x_n^*)\Delta x$, which,

putting in the value of Δx and the x_i^* values, is:

$$f\left(-1+\tfrac{3}{n}\right)\left(\tfrac{3}{n}\right) + f\left(-1+\tfrac{6}{n}\right)\left(\tfrac{3}{n}\right) + \ldots + f\left(-1+\tfrac{3i}{n}\right)\left(\tfrac{3}{n}\right) + \ldots + f\left(-1+\tfrac{3n}{n}\right)\left(\tfrac{3}{n}\right).$$

The picture is:



where the Riemann Sum is the sum of the "areas," area being understood to be negative when the graph is below the x-axis.

Now for the algebra!

Factoring out $\frac{3}{n}$ and evaluating the function produces the form

$$\left(\frac{3}{n}\right)\left\{\left(-1+\frac{3}{n}\right)^3-4\left(-1+\frac{3}{n}\right)+\left(-1+\frac{6}{n}\right)^3-4\left(-1+\frac{6}{n}\right)+\ \dots\right.$$
$$\left.+\left(-1+\frac{3i}{n}\right)^3-4\left(-1+\frac{3i}{n}\right)+\ \dots\ +\left(-1+\frac{3n}{n}\right)^3-4\left(-1+\frac{3n}{n}\right)\right\}=$$
$$\left(\frac{3}{n}\right)\{[(-1+\frac{3}{n})^3+(-1+\frac{6}{n})^3+\ \dots\ +(-1+\frac{3i}{n})^3+\ \dots\ +(-1+\frac{3n}{n})^3]-$$
$$4[(-1+\frac{3}{n})+(-1+\frac{6}{n})+\ \dots\ +(-1+\frac{3i}{n})+\ \dots\ +(-1+\frac{3n}{n})]\}$$

for the Riemann Sum. Let's focus on the parts of this sum:

a.

$$(-1+\frac{3}{n})+(-1+\frac{6}{n})+\ \dots\ +(-1+\frac{3i}{n})+\ \dots\ +(-1+\frac{3n}{n})=-n+\frac{3}{n}(1+2+\ \dots\ +\text{i}+\dots+\text{n})=$$
$$-n+\frac{3}{n}\left(\frac{n(n+1)}{2}\right)=-n+3\left(\frac{(n+1)}{2}\right)=\frac{(n+3)}{2}$$

b.

$$(-1+\frac{3}{n})^3+(-1+\frac{6}{n})^3+\ \dots\ +(-1+\frac{3i}{n})^3+\ \dots\ +(-1+\frac{3n}{n})^3=$$
$$(-1+3(\frac{3}{n})-3(\frac{3}{n})^2+(\frac{3}{n})^3)+(-1+3(\frac{6}{n})-3(\frac{6}{n})^2+(\frac{6}{n})^3)+\ \dots\ +(-1+3(\frac{3i}{n})-3(\frac{3i}{n})^2+(\frac{3i}{n})^3)$$
$$+\ \dots\ +(-1+3(\frac{3n}{n})-3(\frac{3n}{n})^2+(\frac{3n}{n})^3)=-n+3(\frac{3}{n})(1+2+\ \dots\ +\text{i}+\dots+\text{n})\text{-}3(\frac{3}{n})^2(1^2+2^2+\ \dots\ +\text{i}^2+\dots+\text{n}^2)+$$
$$(\frac{3}{n})^3(1^3+2^3+\ \dots\ +\text{i}^3+\dots+\text{n}^3)$$

c. Using the summation formulas:

$$1+2+\ \dots\ +\text{i}+\dots+\text{n}=\frac{n(n+1)}{2}$$
$$1^2+2^2+\ \dots\ +\text{i}^2+\dots+\text{n}^2=\frac{n(n+1)(2n+1)}{6}$$

and $1^3+2^3+\ \dots\ +\text{i}^3+\dots+\text{n}^3=(\frac{n(n+1)}{2})^2$ the expression in part b becomes

$$-n+3(\frac{3}{n})(1+2+\ \dots\ +\text{i}+\dots+\text{n})\text{-}3(\frac{3}{n})^2(1^2+2^2+\ \dots\ +\text{i}^2+\dots+\text{n}^2)+(\frac{3}{n})^3(1^3+2^3+\ \dots\ +\text{i}^3+\dots+\text{n}^3)=$$
$$-n+3(\frac{3}{n})\frac{n(n+1)}{2}\text{-}3(\frac{3}{n})^2\frac{n(n+1)(2n+1)}{6}+(\frac{3}{n})^3(\frac{n(n+1)}{2})^2=$$
$$-n+\frac{9(n+1)}{2}\text{-}\frac{27(n+1)(2n+1)}{6n}+(\frac{27(n+1)^2}{4n})=-n+\frac{9(n+1)}{2}\{1\text{-}\frac{(2n+1)}{n}+\frac{3(n+1)}{2n}\}=$$
$$-n+\frac{9(n+1)}{2}\{\frac{2n-2(2n+1)+3(n+1)}{2n}\}=-n+\frac{9(n+1)}{2}\{\frac{n+1)}{2n}\}=-n+\frac{9(n+1)^2}{4n}=$$
$$\frac{-4n^2+9(n+1)^2}{4n}=\frac{5n^2+18n+9}{4n}$$

d. Putting the results of part a and part c into the Riemann Sum:

Riemann Sum = $\begin{array}{l}(\frac{3}{n})\{[(-1+\frac{3}{n})^3+(-1+\frac{6}{n})^3+\ ...\ +(-1+\frac{3i}{n})^3+\ ...\ +(-1+\frac{3n}{n})^3]-\\ 4[(-1+\frac{3}{n})+(-1+\frac{6}{n})+\ ...\ +(-1+\frac{3i}{n})+\ ...\ +(-1+\frac{3n}{n})]\}\end{array}$ =

$(\frac{3}{n})\{[\frac{5n^2+18n+9}{4n}]-4[\frac{(n+3)}{2}]\}=(\frac{3}{n})\{[\frac{5n^2+18n+9}{4n}]-[\frac{8n(n+3)}{4n}]\}=$
$(\frac{3}{n})[\frac{-3n^2-6n+9}{4n}]=[\frac{-9n^2-18n+27}{4n^2}]=\frac{-9}{4}-\frac{9}{2n}+\frac{27}{4n^2}$, which can also be written as $\frac{-9}{4}-\frac{3}{2}\Delta x+\frac{3}{4}(\Delta x)^2$ by using the relationship $\Delta x=\frac{3}{n}$

5. The last step in determining the value of the integral is to take the limit of the Riemann Sum:

$$\int_{-1}^{2}(x^3-4x)dx=\lim_{\Delta x\to 0}(\frac{-9}{4}-\frac{3}{2}\Delta x+\frac{3}{4}(\Delta x)^2)=\frac{-9}{4}$$, or

$$\int_{-1}^{2}(x^3-4x)dx=\lim_{n\to\infty}(\frac{-9}{4}-\frac{9}{2n}+\frac{27}{4n^2})=\frac{-9}{4}$$

With the integral exactly calculated at this point, I'll show, on the next page, how the Riemann Sum determination would look using the sigma notation:

Riemann Sum =

$$\left(\frac{3}{n}\right)\left\{\sum_{i=1}^{n}[(-1+\frac{3i}{n})^3-4\sum_{i=1}^{n}(-1+\frac{3i}{n})]=\left(\frac{3}{n}\right)\left\{\sum_{i=1}^{n}[(-1)^3+3(\frac{3i}{n})-3(\frac{3i}{n})^2+(\frac{3i}{n})^3+4-\frac{12i}{n}]\right\}=\right.$$

$$\left(\frac{3}{n}\right)\left\{\sum_{i=1}^{n}[3-(\frac{3i}{n})-3(\frac{3i}{n})^2+(\frac{3i}{n})^3]\right\}=\left(\frac{3}{n}\right)\left\{\sum_{i=1}^{n}3-\sum_{i=1}^{n}\frac{3i}{n}-\sum_{i=1}^{n}\frac{27i^2}{n^2}+\sum_{i=1}^{n}\frac{27i^3}{n^3}\right\}=$$

$$\left(\frac{3}{n}\right)\left\{3\sum_{i=1}^{n}1-\frac{3}{n}\sum_{i=1}^{n}i-\frac{27}{n^2}\sum_{i=1}^{n}i^2+\frac{27}{n^3}\sum_{i=1}^{n}i^3\right\}=\left(\frac{3}{n}\right)\left\{3n-\frac{3}{n}\frac{n(n+1)}{2}-\frac{27}{n^2}\frac{n(n+1)(2n+1)}{6}+\frac{27}{n^3}(\frac{n(n+1)}{2})^2\right\}$$

$$9-\frac{9}{2}\frac{(n+1)}{n}-\frac{27}{2}\frac{(n+1)(2n+1)}{n^2}+\frac{81}{4}\frac{(n+1)^2}{n^2}=9-\frac{9}{2}-\frac{9}{2n}-\frac{27}{2}\frac{(2n^2+3n+1)}{n^2}+\frac{81}{4}\frac{(n^2+2n+1)}{n^2}=$$

$$9-\frac{9}{2}-\frac{9}{2n}-27-\frac{81}{2n}-\frac{27}{2n^2}+\frac{81}{4}+\frac{81}{2n}+\frac{81}{4n^2}=9-\frac{9}{2}-\frac{9}{2n}-27-\frac{81}{2n}-\frac{27}{2n^2}+\frac{81}{4}+\frac{81}{2n}+\frac{81}{4n^2}$$

$$=-\frac{9}{4}-\frac{9}{2n}+\frac{27}{4n^2}$$

and we're back to the point reached without using the sigma notation.