Limit Proofs
Hint for Oct 17 Limit
Math 101 – 01

In class 17, Oct 17, the following limit was determined by using the limit theorems:

$\lim_{h\to 0}[\frac{(2+h)^3-2^3}{h}]=12$, and we observed that it follows that if $f(x)=x^3$, then $f'(2)=12$. The following are hints and an outline to prove that $\lim_{h\to 0}[\frac{(2+h)^3-2^3}{h}]=12$ by using the precise definition of limit.

1. Show that the problem reduces to showing that $\lim_{h\to 0}[\frac{12h+6h^2+h^3}{h}]=12$ just by rewriting the function algebraically.
2. Argue that the definition of limit implies that the function $\frac{12h+6h^2+h^3}{h}$ can be replaced by $12+6h+h^2$ and showing $\lim_{h\to 0}[12+6h+h^2]=12$ implies that $\lim_{h\to 0}[\frac{12h+6h^2+h^3}{h}]=12$. (Be sure to state both how the replacement function and original function compare and the critical part of the definition of limit that allows this substitution of a different function.)
3. Show that to prove $\lim_{h\to 0}[12+6h+h^2]=12$ we need, for any given number $\varepsilon>0$, to find a positive number δ, such that whenever $0<|h|<\delta$, then $|6h+h^2|<\varepsilon$. (Here, recall that ε represents the "radius" of the arbitrary interval about 12 on the y-axis and δ represents the "radius" of the sufficiently small interval about 0 on the x-axis. You should have in your mind that there is a "no matter how small" for the arbitrary interval and that the "sufficiently small" means that δ probably (here, definitely) depends on ε. Roughly speaking, the smaller the ε-interval is about 12, the smaller you must make the δ-interval about 0. A picture is essential!)
4. Steps 1-3 come from following the definition of limit carefully. This step, step 4, requires a little extra:
   a. Determine δ_1 so that whenever $0<|h|<\delta_1$, then $|6h|<\frac{\varepsilon}{2}$.
   b. Determine δ_2 so that whenever $0<|h|<\delta_2$, $|h^2|<\frac{\varepsilon}{2}$.

c. Let δ be the smaller of δ_1 and δ_2 and show that whenever $0 < |h| < \delta$, then $|6h| + |h^2| < \varepsilon$.

d. Finally (!), use the triangle inequality for absolute value to show that whenever $0 < |h| < \delta$, then $|6h + h^2| < \varepsilon$. (The triangle inequality says that for any numbers a and b, $|a+b| \le |a| + |b|$.