Calculus I-01
Exam 3 Solutions

Write solutions completely for maximum credit. Put all work on separate pages and <u>keep</u> this page for your reference. Each of the four problems is worth 10 points.

1. Differentiate each of the following with respect to x:

a. xe^{-x^2} b. $\frac{\ln(2x+1)}{\sqrt{2x+1}}$

a. By the product rule, $\frac{d(xe^{-x^2})}{dx} = x\frac{d}{dx}e^{-x^2} + e^{-x^2}\frac{d(x)}{dx}$, and since $\frac{d(x)}{dx} = 1$ and $\frac{d}{dx}e^{-x^2} = e^{-x^2}(\frac{d}{dx}(-x^2)) = e^{-x^2}(-2x)$, we get $\frac{d(xe^{-x^2})}{dx} = xe^{-x^2}(-2x) + e^{-x^2} = e^{-x^2}(-2x^2+1)$

b. By the quotient rule,

$$\frac{d}{dx}\frac{\ln(2x+1)}{\sqrt{2x+1}} = \frac{\sqrt{2x+1}(\frac{2}{2x+1}) - \ln(2x+1)(2)(\frac{1}{2})(2x+1)^{-1/2}}{2x+1}$$

$$= \frac{\sqrt{2x+1}(\frac{2}{2x+1}) - \ln(2x+1)(2)(\frac{1}{2})(2x+1)^{-1/2}}{2x+1} =$$

$$\frac{(\frac{2}{\sqrt{2x+1}}) - \frac{\ln(2x+1)}{\sqrt{2x+1}}}{2x+1} = (\frac{2-\ln(2x+1)}{(2x+1)\sqrt{2x+1}})$$

2. Integrate each of the following with respect to x:

a. $\int \frac{\cos(x/2)}{1+\sin(x/2)}dx$, making the substitution $u = 1+\sin(x/2)$

Let $u = 1 + \sin(x/2)$. Then $du = \frac{1}{2}\cos(x/2)dx$ and $2du = \cos(x/2)dx$.
Substituting into the integral,

$$\int \frac{\cos(x/2)}{1+\sin(x/2)}dx = \int \frac{2}{u}du = 2\ln|u| + c = 2\ln|1+\sin(x/2)| + c$$

To confirm the solution, differentiate the answer using the rule for constant multiples and the chain rule twice to get:

$$\frac{d}{dx}(2\ln|1+\sin(x/2)|) = 2\frac{d}{dx}(\ln|1+\sin(x/2)|) =$$
$$2\frac{1}{1+\sin(x/2)}\frac{d}{dx}(1+\sin(x/2)) = 2\frac{1}{1+\sin(x/2)}\cos(x/2)\frac{d}{dx}(x/2) =$$
$$2\frac{1}{1+\sin(x/2)}\cos(x/2)(1/2) = \frac{\cos(x/2)}{1+\sin(x/2)}$$

b. $\int \frac{1}{(2x+1)\ln(2x+1)}dx$, making the substitution $u = \ln(2x+1)$

Let $u = \ln(2x+1)$. Then $du = \frac{1}{2x+1}(2)dx$ and $\frac{1}{2}du = \frac{1}{2x+1}dx$.
Substituting into the integral,

$$\int \frac{1}{(2x+1)\ln(2x+1)}dx = \int \frac{1}{2}\frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|\ln(2x+1)| + c$$

To confirm the solution, differentiate the answer using the rule for constant multiples and the chain rule three times to get:

$$\frac{d}{dx}(\frac{1}{2}\ln|\ln(2x+1)|) = \frac{1}{2}\frac{d}{dx}(\ln|\ln(2x+1)|) = \frac{1}{2}\frac{1}{\ln(2x+1)}\frac{d}{dx}(\ln(2x+1)) =$$
$$\frac{1}{2}\frac{1}{\ln(2x+1)}\frac{1}{2x+1}\frac{d}{dx}(2x+1) = \frac{1}{2}\frac{1}{\ln(2x+1)}\frac{1}{2x+1}(2) = \frac{1}{\ln(2x+1)}\frac{1}{2x+1}$$

3. Suppose that f is a function defined by $f(x) = \frac{4}{1+2\ln x}$, and assume that f is 1-1.

a. Determine $f'(x)$.

Using the quotient rule, the rule for constants, and the rule for sums we get:

$$f'(x) = \frac{(1+2\ln x)\frac{d}{dx}(4) - 4\frac{d}{dx}(1+2\ln x)}{(1+2\ln x)^2} = \frac{-4(\frac{2}{x})}{(1+2\ln x)^2} = \frac{-8}{x(1+2\ln x)^2}$$. Writing f(x) as $4(1+2\ln x)^{-1}$, and using the rule for constant multiples, the rule for sums, and the chain rule also works well.

b. Determine a formula for $f^{-1}(x)$

Start by writing $y = \frac{4}{1+2\ln x}$ and solve for x:

$$y(1+2\ln x) = 4 \Rightarrow y + 2y\ln x = 4 \Rightarrow 2y\ln x = 4 - y \Rightarrow \ln x = \frac{4}{2y} - \frac{y}{2y} = \frac{2}{y} - \frac{1}{2}$$

$\Rightarrow x = e^{(\frac{2}{y}-\frac{1}{2})}$, by using the definition of ln.

Then replace x by y and y by x to get: $f^{-1}(x) = e^{(\frac{2}{x}-\frac{1}{2})}$

4. Assume that the equation $xe^y + y = 1$ defines y implicitly as a function of x.

a. Determine the value of $\frac{dy}{dx}$ at the point (1,0).

Differentiating each side of the equation and using various rules of differentiation, we have:

$$xe^y + y = 1 \Rightarrow \frac{d}{dx}(xe^y + y) = \frac{d}{dx}(1) \Rightarrow \frac{d}{dx}(xe^y) + \frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{d}{dx}(e^y) + e^y\frac{d}{dx}(x) + \frac{dy}{dx} = 0 \Rightarrow xe^y\frac{dy}{dx} + e^y + \frac{dy}{dx} = 0$$

$$\Rightarrow (xe^y + 1)\frac{dy}{dx} = -e^y \Rightarrow \frac{dy}{dx} = \frac{-e^y}{(xe^y+1)}$$. At the point (1,0), where x=1 and y=0, $\frac{dy}{dx} = \frac{-e^0}{((1)e^0+1)} = \frac{-1}{2}$.

b. Determine where the tangent line to the graph of the equation $xe^y + y = 1$ at the point (1,0) intersects the y-axis. (If you do not get an answer to part a, state an assumption for the answer to part a in order to answer this part of the problem.)

Since the tangent line passes through the point (1,0), if you go to the left 1 unit from (1,0) to get to the y-axis, you would need then to go up 1/2 unit to get to a point on the y-axis that is on the tangent line, since the slope of the tangent line is –1/2.