Calculus I-01
Exam 1 – Team Portion

1. Consider the function defined by $k(x)=\sqrt{x}$.

a. Determine the average rate of change in k over the interval [9,16].

The average rate of change in k over the interval [9,16] is:

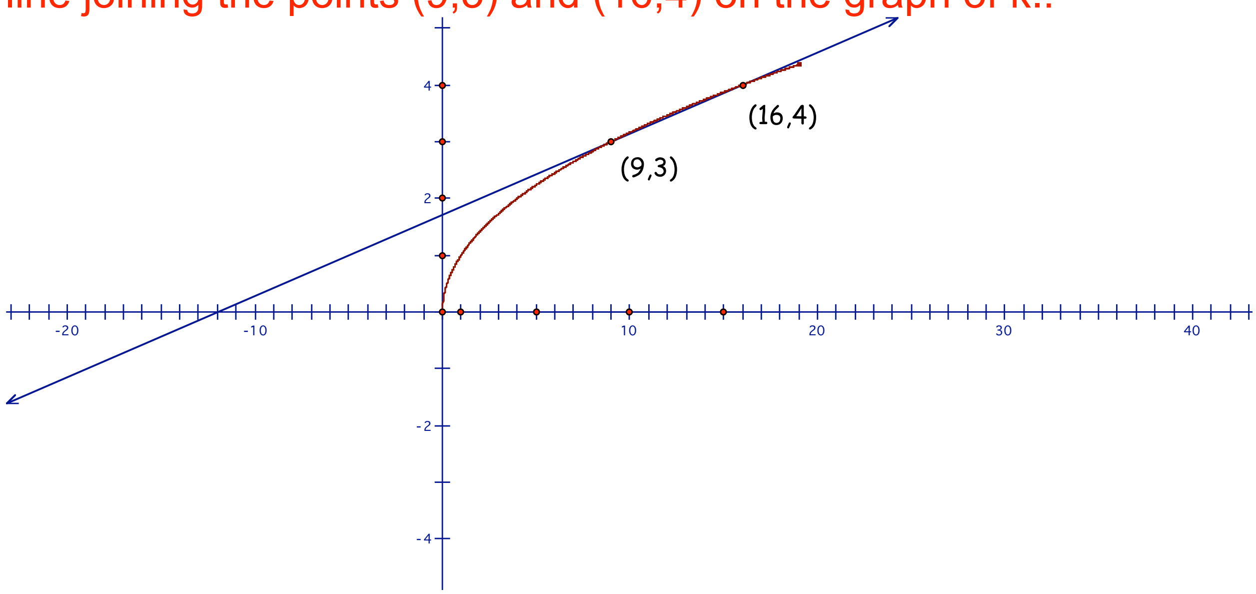
$$\frac{k(16)-k(9)}{16-9}=\frac{\sqrt{16}-\sqrt{9}}{7}=\frac{4-3}{7}=\frac{1}{7}.$$

b. For the function g defined in #2 below and the function k above, determine a formula for $(g\circ k)(x)$ and the domain of $g\circ k$.

$(g\circ k)(x)=g(k(x))=g(\sqrt{x})=\frac{3}{2\sqrt{x}}$. The function $g\circ k$ is defined only for x > 0, so its domain is the infinite open interval $(0,\infty)$.

c. Draw a rough sketch of the graph of k and a line passing through two points on the graph of k whose slope is the same as the average rate of change in k over the interval [9,16].

The average rate of change in k over the interval [9,16] (the interval so that $9\le x\le 16$) is the same as the slope of the secant line joining the points (9,3) and (16,4) on the graph of k.:



d. Determine the slope of the tangent line to the graph of k at the point (9,3).

The slope of the tangent line to the graph of k at (9,3) is k'(9).

$$k'(9) = \lim_{h\to 0}\frac{k(9+h)-k(9)}{(9+h)-9} = \lim_{h\to 0}\frac{\sqrt{9+h}-\sqrt{9}}{h} = \lim_{h\to 0}\frac{\sqrt{9+h}-3}{h} =$$

$$\lim_{h\to 0}\left(\frac{\sqrt{9+h}-3}{h}\right)\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right) = \lim_{h\to 0}\left(\frac{9+h-9}{h(\sqrt{9+h}+3)}\right) = \lim_{h\to 0}\left(\frac{1}{(\sqrt{9+h}+3)}\right) =$$

$$\frac{1}{(\sqrt{9}+3)} = \frac{1}{6}$$

Comments and Additional Questions/Problems: An unfortunately common mistake in this problem is to interpret [9,16] as a point on the graph of the function k.

P1.1. Explain why – in addition to being described in the problem as an interval and written with square brackets, rather than parentheses - [9,16] could not be a point on the graph of the function k.

P1.2. Describe, using the derivative concept and geometric terms, the steepness of the graph of k at the origin.

P1.3. Suppose that (p,q), with p>0, is any point on the graph of the function k. Determine an equation for the tangent line to the graph of k at (p,q). Then determine where, if anywhere, this tangent line intersects the x-axis.

2. Consider the function defined by $g(x)=\frac{3}{2x}$.

a. Evaluate the expressions $g(1)$ and $g(1+h)$.

$g(1)=\frac{3}{2(1)}=\frac{3}{2}$ and $g(1+h)=\frac{3}{2(1+h)}$.

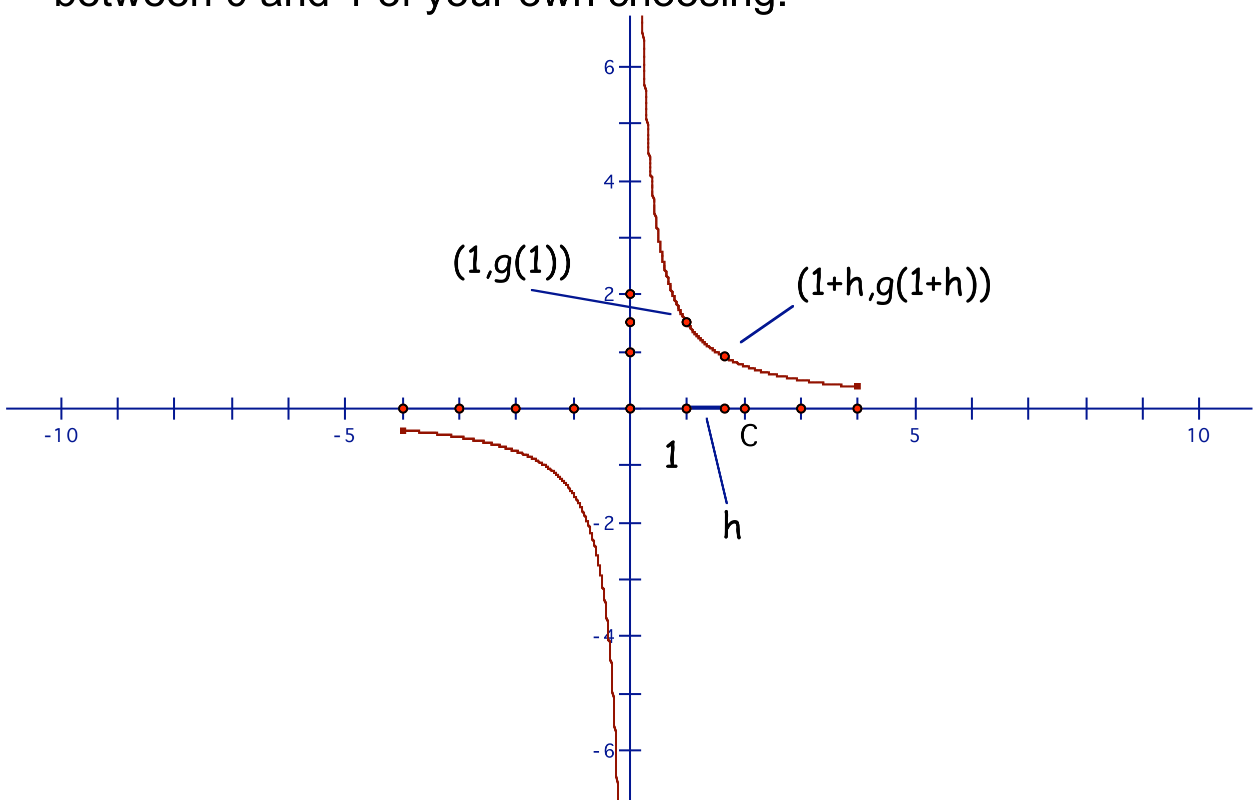
b. Evaluate the expression $\frac{g(1+h)-g(1)}{h}$ and give a brief description of what it represents.

$$\frac{g(1+h)-g(1)}{h}=\frac{\frac{3}{2(1+h)}-\frac{3}{2}}{h}=\frac{\frac{3-(3)(1+h)}{2(1+h)}}{h}=\frac{3-3-3h}{2(1+h)h}$$

$$=\frac{-3h}{2(1+h)h}$$

This difference quotient represents the average rate of change in the function g over the interval [1,1+h] (or [1+h,1], in the case that h is negative.) It also represents the slope of the secant line joining the points (1,3/2) and (1+h, 3/[2+2h]) on the graph of g.

c. Draw a rough sketch of the graph of g and locate the points (1,g(1)) and ((1+h), g(1+h)) on the graph for some value of h between 0 and 1 of your own choosing.



d. Determine $\lim_{h\to 0}[\frac{g(1+h)-g(1)}{h}]$, or show that this limit does not exist.

$$\lim_{h\to 0}[\frac{g(1+h)-g(1)}{h}]=\lim_{h\to 0}[\frac{-3h}{2(1+h)h}]=\lim_{h\to 0}[\frac{-3}{2(1+h)}]=\frac{-3}{2}$$

Comments and Additional Questions/Problems: A common mistake in this problem is to think the algebraic portion of the answer to part b represents the slope of a tangent line to the graph of g.

P2.1. Explain why the algebraic portion of the answer to part b does not represent the slope of a tangent line to the graph of g.

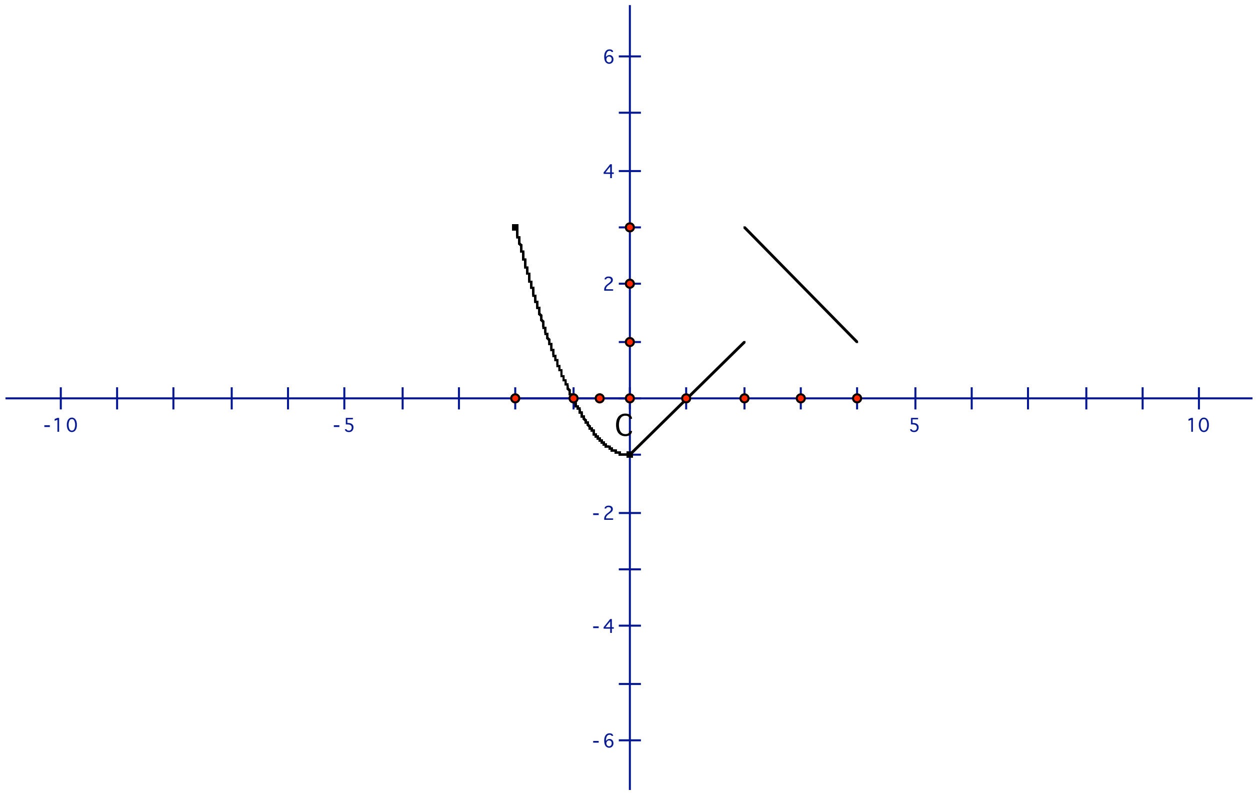
P2.2. Explain why, in geometric terms, the algebraic portion of the answer to part b should be dependent on h, but the answer to part d should not depend on h.

P2.3. Explain very precisely what the answer to part d represents.

3. Consider the function defined by:

$$f(x)=\begin{cases} x^2-1, \text{ if } -2<x\le 0 \\ x-1, \text{ if } 0<x\le 2 \\ 5-x, \text{ if } 2<x\le 4 \end{cases}$$

a. Draw the graph and determine the domain of the function f.
The function f is defined for $-2<x\le 4$ so the domain of f is the half-open interval (-2,4]. The graph, consisting of 3 “pieces,” a parabolic arc and two line segments, appears below. Note that f is not defined at x=-2, which is often depicted by placing a small open circle at the point (-2,3).



a. Determine $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 2^-} f(x)$, if they exist.

From the graph, it is clear that $\lim_{x \to 0^+} f(x)$ is –1 and that $\lim_{x \to 2^-} f(x)$ is 1.

a. Determine any/all numbers in the domain of f where f is discontinuous.

Again, from the graph, it is clear that the only value of x in the domain of f for which f is discontinuous ('broken' graph) is the number 2.

a. Determine the average rates of change in f over each of the following three intervals: [-0.1,0], [-0.01,0], [0,0.01]

The average rate of change in f over the interval [-0.1,0] is:

$\frac{f(0)-f(-0.1)}{0-(-0.1)} = \frac{(0^2-1)-([-0.1]^2-1)}{0.1} = \frac{-1-(0.01-1)}{0.1} = \frac{-0.01}{0.1} = -0.1$, over [-0.01,0]:

$\frac{f(0)-f(-0.01)}{0-(-0.01)} = \frac{(0^2-1)-([-0.01]^2-1)}{0.01} = \frac{-1-(0.0001-1)}{0.01} = \frac{-0.0001}{0.01} = -0.01$, and over

[0,0.01]: $\frac{f(0)-f(0.01)}{0-(0.01)} = \frac{(0^2-1)-(0.01-1)}{-0.01} = \frac{-0.01}{-0.01} = 1$

C<u>omments and Additional Questions/Problems</u>: A common mistake in this problem is to think that the function f is not defined at x=0 or that it is not continuous at x=0, possibly because the value of the function when x=0 is not determined by the first of the 3 formulas that define the one function f. Another common mistake is to include –2 as an answer to part c.

P3.1. Explain why –2 is not an answer to part c.

P3.2. Suppose that p is a "small" positive number, less than 1. Determine a. the average rate of change in f over the interval [0,p] and b. the average rate of change in f over the interval [-p,0].

P3.3. Explain your answers to P3.2 in terms of the derivative concept.

4. Suppose that the manufacturer of *jPods* collects data to show that the total revenue produced from the sale of 20,000 *jPods* is $300,000 and from the sale of 40,000 *jPods* is $620,000.

a. What total revenue could the manufacturer expect when selling 45,000 *jPods*, assuming that a <u>linear</u> model for the relation between total revenue and number of *jPods* sold is appropriate.
We can construct a function f for the total revenue, in dollars, f(x) produced from the sale of x *jPods* . Then (20,000, 300,000) and (40,000, 620,000) are points on the graph of f and, since the relationship is assumed to be linear, the graph will have a constant slope $\frac{620000-300000}{40000-20000}=\frac{320000}{20000}=16$. Using the point-slope form for a line, $f(x)=16(x-20000)+300000$. The total revenue, in dollars, when 45,000 *jPods* are sold is then
$f(45000)=16(45000-20000)+300000=$
$16(25000)+300000=400000+300000=700000$

a. Determine the average revenue per *jPod* sold over the interval 20,000 *jPods* to 40,000 *jPods* and the average revenue per *jPod* over the interval 40,000 *jPods* to 45,000 *jPods*.

From part a, the average revenue per *jPod* sold is the constant slope 16, which means \$16 per *jPod*.

a. Suppose that, instead of a linear model, the manufacturer chooses and exponential model, $R(x) = 10{,}000(2^{2.5x} - 2)$, where x represents the number of tens of thousands of *jPods* sold and R(x) represents the revenue in dollars. Show that R(2)=300,000 and determine the revenue when 40,000 *jPods* are sold and when 45,000 *jPods* are sold.

$R(2) = 10{,}000(2^5 - 2) = 10{,}000(30) = 300{,}000$. The figure of 40,000 *jPods* means that x=4, so $R(4) = 10{,}000(2^{10} - 2) = 10{,}000(1022) = 10{,}220{,}000$ is the revenue, in dollars, when 40,000 *jPods* are sold. Similarly, the revenue, in dollars, when 45,000 *jPods* are sold is $R(4.5) = 10{,}000(2^{45/4} - 2) \approx 10{,}000(2435.5 - 2) = 24{,}335{,}000$.

a. For the exponential model in part c, determine the average revenue per *jPod* sold over the interval 20,000 *jPods* to 40,000 *jPods* and the average revenue per *jPod* over the interval 40,000 *jPods* to 45,000 *jPods*.

The average revenue per *jPod* sold over the interval 20,000 *jPods* to 40,000 *jPods* : First the average per 10,000 *jPods* is $\frac{R(4) - R(2)}{4 - 2} = \frac{10{,}220{,}000 - 300{,}000}{4 - 2} = \frac{9{,}920{,}000}{2} = 4{,}960{,}000$, so the average revenue per *jPod* is \$496 per *jPod* over the interval 20,000 *jPods* to 40,000 *jPods* . Similarly, The average revenue per *jPod* sold over the interval 40,000 *jPods* to 45,000 *jPods* is: $\frac{R(4.5) - R(4)}{10{,}000(4.5 - 4)} = \frac{24{,}335{,}000 - 10{,}220{,}000}{10{,}000(0.5)} = \frac{14{,}115{,}000}{5{,}000} = 2{,}823$; i.e., the average revenue per *jPod* is \$2,823 per *jPod* over the interval 40,000 *jPods* to 45,000 *jPods* .

Comments and Additional Questions/Problems: The most common error in proposed solutions to this problem is to assume that the total revenue is zero when the number of *jPods* sold is zero. But such an assumption is not consistent with the data and having a linear model.

P4.1. Explain the inconsistency between:

a. The graph of revenue vs. number *jPods* sold goes through the origin

and

b. The given data points and the linear model.

P4.2. Give a plausible explanation (in real-world terms) for the fact that f(0)=-20,000 <u>or</u> explain why you think this particular linear model for the data does not fit a real-world situation.

P4.3. Explain why you think the particular exponential model described in part c fits, or does not fit, a real-world situation.