1 Statement of Problem

A conducting disk is contacted by an AFM tip at potential \( \phi = 1 \) at the center of one face, the other face being grounded. The disk has radius \( R = 1 \), height \( h \), that is, in cylindrical coordinates \((\rho, \theta, z)\) it occupies the region \( D \) given by \( \rho \leq 1, \, 0 \leq z \leq h \). The AFM tip contact is the centered circular disk \( \rho \leq a, \, z = h \). Let the conductivity of the disk be \( \sigma \). Then the steady electrostatic potential \( \phi \) is to be found, and the current density in the disk is \( \vec{j} = -\sigma \vec{\nabla} \phi \).

The potential \( \phi \) is a harmonic function satisfying mixed boundary conditions

\[
\begin{align*}
\nabla^2 \phi &= 0 \quad \text{in} \ D & (1) \\
\phi &= 0, \quad (z = 0) & (2) \\
\frac{\partial \phi}{\partial \rho} &= 0, \quad (\rho = 1) & (3) \\
\phi &= 1, \quad (\rho \leq a, \, z = h) & (4) \\
\frac{\partial \phi}{\partial z} &= 0, \quad (\rho > a, \, z = h) & (5)
\end{align*}
\]

The boundary conditions express the imposed values of \( \phi \) at some places and the condition that no current passes through the boundary at others, equivalent to taking the conductivity of the (gold) substrate to be infinite and the conductivity of the air to be zero.

2 Solution for Current

By standard reasoning, the solution for \( \phi \) can be expressed in \( L^2(D) \) as

\[
\phi = \sum_{n=0}^{\infty} A_n f_n(\rho) g_n(z)
\]

where the basis functions are chosen (normalized) to be

\[
\begin{align*}
 f_n(\rho) &= \frac{\sqrt{2}}{|J_0(\alpha_n)|} J_0(\alpha_n \rho), \quad (n = 0, 1, \ldots) & (7) \\
 g_0(z) &= z/h & (8) \\
 g_n(z) &= \sinh(\alpha_n z), \quad (n = 1, 2, \ldots) & (9)
\end{align*}
\]
and $\alpha_n$ is the $n$th zero of the Bessel function $J_1$, including $\alpha_0 = 0$. (Note: $f_0 = \sqrt{2}$, a constant function.) This expression for $\phi$ already satisfies Eqs. (1)-(3). It remains to choose the $A_n$ to satisfy Eqs. (4)-(5). The boundary conditions require

\begin{align}
1 &= A_0 \sqrt{2} + \sum_{n=1} A_n \frac{\sqrt{2} J_0(\alpha_n \rho)}{|J_0(\alpha_n)|} \quad (\rho \leq a) \\
0 &= A_0 \sqrt{2} + \sum_{n=1} A_n \frac{\sqrt{2} J_0(\alpha_n \rho)}{|J_0(\alpha_n)|} \quad (\rho > a).
\end{align}

The basis functions $f_n$ have been normalized so that

\begin{equation}
\int_0^1 f_m(\rho) f_n(\rho) \rho d\rho = \delta_{mn},
\end{equation}

but to make use of the conditions in Eqs. (10)-(11) we also need the integrals

\begin{align}
K_{mn} &= \int_0^a f_m(\rho) f_n(\rho) \rho d\rho, \\
L_{mn} &= \int_a^1 f_m(\rho) f_n(\rho) \rho d\rho = \delta_{mn} - K_{mn},
\end{align}

defining matrices that we will simply call $K$ and $L = I - K$. By standard properties of Bessel functions we have the explicit formulae

\begin{align}
K_{mn} &= \frac{2a[\alpha_m J_1(\alpha_n a) J_0(\alpha_m a) - \alpha_n J_0(\alpha_n a) J_1(\alpha_m a)]}{(\alpha_m^2 - \alpha_n^2)|J_0(\alpha_m)|J_0(\alpha_n)|} \quad (m \neq n) \\
K_{nn} &= \frac{a^2 [J_0(\alpha_n a)^2 + J_1(\alpha_n a)^2]}{|J_0(\alpha_n)|^2};
\end{align}

Anticipating that we will truncate the sum in Eq. (6) at $n = N$, let us also define diagonal matrices

\begin{align}
S &= \text{diag}(1, \sinh(\alpha_1 h), ..., \sinh(\alpha_N h)) \\
C &= \text{diag}(1, \alpha_1 h \cosh(\alpha_1 h), ..., \alpha_N h \cosh(\alpha_N h)).
\end{align}

Regard the coefficients $\{A_n\}$ as the column vector $\vec{A}$ and the zeroth column of the matrix $K$ as the vector $\vec{K}_0$. Then the boundary conditions imply

\begin{align}
S \vec{A} &= \frac{\vec{K}_0}{\sqrt{2}} + LS \vec{A} \\
C \vec{A} &= K C \vec{A}
\end{align}

or, more simply,

\begin{align}
KS \vec{A} &= \frac{\vec{K}_0}{\sqrt{2}} \\
LC \vec{A} &= 0
\end{align}
The integral operators underlying the truncated versions $K$ and $L$ have infinite dimensional null spaces, namely all functions that vanish on $[0, a]$, or on $[a, 1]$. This gives Eqs. (21)-(22) a simple and clear meaning, but suggests trouble for numerical methods. It is clear, for example, that $\phi(\rho, h)$ is not analytic at $\rho = a$. The best method is probably to solve in the sense of least squares, since convergence in Eq. (6) is only guaranteed in $L^2(D)$. The results behave as expected: slow convergence of $\phi(\rho, h)$ near $\rho = a$, and Stokes phenomenon in $\partial \phi / \partial z$ at $\rho = a$. On the other hand, convergence is excellent at $z = 0$, and integrating $j \cdot \hat{n}$ over the base of the cylinder one accurately finds the total current

$$I = \left( \frac{\sqrt{2}\pi A_0}{h} \right) (\sigma V_0 R), \tag{23}$$

where we have restored the dimensional factors $\sigma$ (the conductivity), $V_0$ (the applied voltage), and $R$ (the radius of the disk). One finds that the current $I$ decreases with the contact area $\pi a^2$, and thus the average current density $I/\pi R^2$ at the base decreases, but the average current density in the contact region $I/\pi a^2$ increases. Thus a magnetized ring within the contact area (i.e., of fixed radius less than $a$) would feel a stronger magnetic field in the smaller contact area. Dimensionless values for $h = 0.05$ are shown in the table:

<table>
<thead>
<tr>
<th>a</th>
<th>$I/\pi a^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>65.2</td>
</tr>
<tr>
<td>0.05</td>
<td>39.1</td>
</tr>
<tr>
<td>0.1</td>
<td>29.0</td>
</tr>
<tr>
<td>0.2</td>
<td>24.2</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

The smallest value of $a$ in the table is still larger than the small values relevant to the experiment, but the numerical method begins to fail here. The reason is that the truncation of the matrix $K$ to $N \times N$ leads to typical problems if $N$ is too large, around $N = 100$, but to capture information around $\rho = a$ requires $N \sim 2/a$. For this purpose it would be useful to find an asymptotic approximation to the theory for small $a$.

If $a$ is small, we can regard $x = n \pi a$ as a continuous variable sampled at integer multiples of $\pi a$. Values like $J_1(\alpha_n a)$ can be thought of as values of $J_1(x)$ sampled at regular intervals, because even small values of $x > 0$ correspond to large values of $\alpha_n$, so that we can take $\alpha_n \sim n \pi$. Then making the replacements $m \pi a \rightarrow y$, $n \pi a \rightarrow x$, we have, for $m \neq 0$, $n \neq 0$,

$$K_{nm} \sim a K(x, y) = a \frac{\pi}{2} \sqrt{xy} \left[ \frac{J_1(y)J_0(x) - J_0(y)J_1(x)}{y - x} + \frac{J_1(y)J_0(x) + J_0(y)J_1(x)}{y + x} \right] \tag{24}$$
Here $K(x, y)$ is of order unity for $x, y > 0$. On the other hand, for $n \neq 0$,

$$K_{0n} \sim K(x) = \frac{2a J_1(\alpha_n a)}{\alpha_n |J_0(\alpha_n)|} \sim \sqrt{2\pi} a^{3/2} J_1(x) \sqrt{x}$$  \tag{25}

and

$$K_{00} = a^2$$  \tag{26}

We also have

$$\sum_{n=1}^{\infty} \cdots \to \frac{1}{\pi a} \int_{0}^{\infty} \cdots dx$$  \tag{27}

The $n = 0$ term in the sums is not represented by the integral and must be put in by hand.

If we now look at Eq. (19), we find for $A_0$

$$A_0 \sim \frac{a^2}{\sqrt{2}} + A_0 - \frac{1}{\pi a} \int_{0}^{\infty} a^{3/2} \sqrt{2\pi} \frac{J_1(x)}{\sqrt{x}} S\vec{A}(x) \, dx$$  \tag{28}

and this tells us nothing about $A_0$, since it cancels out of the equation. (In the integral $S\vec{A}(x)$ is the vector $S\vec{A}$ thought of as a continuous function.) For $x > 0$ we have

$$\frac{1}{\pi} \int_{0}^{\infty} K(x, y) S\vec{A}(y) \, dy \sim a^{3/2} \sqrt{\pi} \frac{J_1(x)}{\sqrt{x}}$$  \tag{29}

Thus $S\vec{A}(y) \sim a^{3/2}$. Define $S\vec{A} = S\vec{A}/a^{3/2}$.

(30)

From Eq. (20),

$$\frac{A_0}{ah} \sim \frac{1}{\pi} \int_{0}^{\infty} \sqrt{2\pi} \frac{J_1(x)}{\sqrt{x}} x \coth(xh/a) S\vec{A}(x) \, dx$$  \tag{31}

In all, by Eqs. (20) and (29), we have

$$0 \sim \frac{1}{\pi} \int_{0}^{\infty} K(x, y) y \coth(yh/a) S\vec{A}(y) \, dy - x S\vec{A}(x)$$  \tag{32}

$$\sqrt{\pi} \frac{J_1(x)}{\sqrt{x}} \sim \frac{1}{\pi} \int_{0}^{\infty} K(x, y) S\vec{A}(y) \, dy,$$  \tag{33}

a system of equations for $S\vec{A}$ that is essentially like Eqs. (19)-(20) but with better numerical properties, and solved the same way. This solution in Eq. (31) determines $A_0/ah$, and hence the total current $I$, as a function of $a/h$. Eq. (28) in the form

$$\frac{1}{\sqrt{2}} \sim \frac{1}{\pi} \int_{0}^{\infty} \sqrt{2\pi} \frac{J_1(x)}{\sqrt{x}} S\vec{A}(x) \, dx$$  \tag{34}

is a check of numerical accuracy of the solution for $S\vec{A}$.

The asymptotic solution agrees very well with the solution found before, at least where it can be calculated, even for $a$ as large as 0.2, which is much
larger than the AFM tip in this experiment. Thus one should simply use the asymptotic solution, given in the table:

<table>
<thead>
<tr>
<th>a/h</th>
<th>( A_0/ah )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.56</td>
</tr>
<tr>
<td>5</td>
<td>4.10</td>
</tr>
<tr>
<td>3</td>
<td>2.73</td>
</tr>
<tr>
<td>2</td>
<td>2.06</td>
</tr>
<tr>
<td>1</td>
<td>1.41</td>
</tr>
<tr>
<td>0.5</td>
<td>1.13</td>
</tr>
<tr>
<td>0.2</td>
<td>1.00</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
</tr>
</tbody>
</table>

For large \( a/h \) (i.e., a very thin disk), using Eq. (23), the solution is essentially \( I \sim V_0 \sigma R \pi a^2 / h \), as if the disk were a very short wire of radius \( a \). For small \( a/h \), the contact is essentially a small conductance in series with an infinite conductance (the wide disk), and \( I \sim \sqrt{2} \pi V_0 \sigma a R \) (\( a R \) is the actual radius of the contact area since \( a \) is the dimensionless radius). The magnetic field \( B_{\text{max}} = B(a) \) at the edge of the contact area, (where \( \rho = a R \) and the field is maximal), decreases with \( a \) down to \( a \sim h \), since the current decreases as \( a^2 \), but for \( a < h \) this field \( B_{\text{max}} \) becomes constant. In fact,

\[
B_{\text{max}} = \frac{\mu_0 I}{2 \pi a R} = \frac{1}{\sqrt{2}} \left( \frac{A_0}{ah} \right) \mu_0 \sigma V_0,
\]

so that except for dimensional factors its dependence on \( a \) is that of \( A_0/ah \) in the table for fixed \( h \). If \( h R = 50 \) nm and \( a R = 10 \) nm, then the tip size is already in this range of constant \( B_{\text{max}} \), so making the tip smaller would not increase \( B_{\text{max}} \). It may be, however, that the same \( B_{\text{max}} \), applied more locally, is more effective in starting the process of reversing the vorticity.

Treating the system as a classical continuum only makes sense if the electron mean free path and the electron wavelength are both much less than \( a R \). That seems to be true in this experiment, although I have not checked this carefully.

3 Solution for Magnetic Field

In this section I consider the distribution of current in more detail, and the resulting magnetic field. By the azimuthal symmetry of the current distribution \( \vec{j} = -\sigma \nabla \phi \), the magnetic field \( \vec{B} \) can still be found by Ampere’s Law: the magnetic field circulates around the azimuthal symmetry axis, and its magnitude is

\[
B(\rho, z) = \frac{\mu_0}{\rho} \int_0^\rho j_{\rho, \rho} d\rho
\]

(36)
where the z-component of dimensionless current (i.e., set $\sigma = 1$) is

$$j_z = \frac{\sqrt{2} A_0}{h} + \sum_{n=1}^{\infty} A_n \alpha_n \cosh(\alpha_n z) \frac{\sqrt{2} J_0(\alpha_n \rho)}{|J_0(\alpha_n)|} \rho d\rho$$  \hspace{1cm} (37)

Using

$$\int_{J_0^2}^{z} J_0(x) x \, dx = z J_1(z), \quad \text{[Gradshteyn and Ryzhik 5.52.1]}$$  \hspace{1cm} (38)

we have

$$\frac{1}{\mu_0} B(\rho, z) = \frac{A_0 \rho}{\sqrt{2} h} + \sum_{n=1}^{\infty} A_n \frac{\sqrt{2} \cosh(\alpha_n z)}{|J_0(\alpha_n)|} J_1(\alpha_n \rho)$$  \hspace{1cm} (39)

Making the asymptotic replacements

$$\alpha_n a \to x$$  \hspace{1cm} (40)

$$\sum_{n=1}^{\infty} \to \frac{1}{\pi a} \int_0^{\infty} ... \, dx$$  \hspace{1cm} (41)

$$A_n \to a^{3/2} \frac{S A(x)}{\sinh(xh/a)}$$  \hspace{1cm} (42)

Eq. (39) becomes asymptotically

$$\frac{1}{\mu_0} B(\rho, z) \sim \frac{A_0 \rho}{\sqrt{2} h} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{S A(x)}{\sinh(xh/a)} \frac{\cosh(\rho x/a)}{\sqrt{x} J_1(\rho x/a)} \, dx$$  \hspace{1cm} (43)