1. The Koebe function is
\[ f(z) = \frac{z}{(1-z)^2} \]  \hspace{1cm} (1)

(a) Find the Taylor series at \( z = 0 \) for the Koebe function \( f \). What is the radius of convergence?

(b) Where, if anywhere, does \( f \) fail to be conformal?

(c) Thinking of the Koebe function \( f \) as a map from the unit disk \( |z| < 1 \) to the complex plane, where does it fail to be one-to-one? Investigate this by looking at the image of the boundary of the unit disk, that is, the unit circle \( |z| = 1 \). What does the image look like?

(d) Investigate the one-to-one property by solving \( f(z) = c \). How many solutions are there in \( |z| < 1 \)? What are the possible values of \( c \)?

(e) An analytic function \( F \) which is one-to-one on the unit disk and normalized so that \( F(0) = 0 \), \( F'(0) = 1 \) is called \textit{schlicht}. Show that the Koebe function is \textit{schlicht}.

(f) The Bieberbach conjecture, a famous unsolved problem through much of the 20th century, stated that the Maclaurin coefficients \( \{a_n\} \) of a \textit{schlicht} function satisfy \( |a_n| \leq n \) for all \( n = 1, 2, 3, ... \) Note why this is interesting: the Maclaurin coefficients seem to express only local properties of \( f \), but the one-to-oneness of \textit{schlicht} functions is \textit{global}. The Bieberbach conjecture was eventually proved in 1985 by de Branges. What does the Koebe function have to do with this conjecture?

2. (a) Find a formula for \( \sin^{-1}(w) \) by solving \( w = \sin(z) \) for \( z \). Hint: represent
\[ \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \]  \hspace{1cm} (2)
(b) Differentiate your formula in (a) with respect to \( w \), and verify that the derivative of the inverse sine is what you learned in beginning calculus:

\[
\frac{d}{dw} \left( \sin^{-1}(w) \right) = \frac{1}{\sqrt{1 - w^2}}.
\]

(3)

(c) Consider the contour integral representation of the inverse sine

\[
\sin^{-1}(w) = \int_C \frac{1}{\sqrt{1 - \omega^2}} d\omega
\]

where \( C \) is a contour which starts at 0 and ends at \( w \). To evaluate such an integral, one must be aware of the analytic structure of the integrand. Discuss this. If you restrict attention to one branch of the integrand, where are the cuts? What does this have to do with the cut in the logarithm function?

(d) The inverse sine function has many branches, that is, it is multivalued, and is not well defined until one makes some decision about choosing one unique value for a given \( w \). Where does this multivaluedness come from in the contour integral representation? Why isn’t the contour integral well defined? Isn’t it true that the contour integral of an analytic function is independent of the contour, and depends only on the endpoints?

3. A sequence of complex numbers \( \{z_n\} \) is Cauchy if for any \( \epsilon > 0 \) there exists \( N \) such that \( |z_n - z_m| < \epsilon \) provided \( n > N, m > N \). That is, if you go out far enough in the sequence, the \( z_n \)'s are all close to each other. The most important consequence is that the sequence has a limit (converges). Use that fact below.

(a) A sequence of functions \( \{f_n\} \) is uniformly Cauchy in \( E \) if for any \( \epsilon > 0 \) there exists \( N \) such that \( |f_n(z) - f_m(z)| < \epsilon \) for all \( z \) in \( E \) provided \( n > N, m > N \). Show that a uniformly Cauchy sequence of functions converges uniformly in \( E \).

(b) Suppose that \( C \) is a simple closed path and let \( \Omega \) denote the region interior to \( C \). Suppose that \( f_n (n = 1, 2, \ldots) \) is analytic on \( \Omega \) and continuous
on $C$. Suppose that $\{f_n\}$ converges uniformly to some function $f$ on $C$. Show that $\{f_n\}$ converges to an analytic function on $\Omega$. [Hint: use the maximum modulus principle.]

(c) Suppose that

$$f(\theta) = \Re \sum_{n=0}^{\infty} a_n e^{in\theta}$$

is the Fourier representation of the real periodic function $f(\theta)$, and that the series $\sum |a_n|$ converges. (Here $\Re$ means “real part.”) Show that

$$u(z) = \Re \sum_{n=0}^{\infty} a_n z^n$$

solves the Dirichlet problem in the disk $|z| \leq 1$ with boundary values $f(\theta)$.

4. (a) Show that a rational function of a complex variable $z$ takes every value (including $\infty$) the same number of times (counting with multiplicity).

(b) Consider in particular the rational functions called *fractional linear transformations*,

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$. How many times does $f(z)$ take each complex value?

(c) Where is the fractional linear transformation conformal?

5. Show that the Riemann zeta function $\zeta(s)$ can be represented by the integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx$$

for $\Re(s) > 1$. [Hint: re-express the denominator as an appropriate geometric series. Recall the gamma function $\Gamma(z)$ from problem 4.3.24 on p. 273.]

6. Be prepared to do contour integrals of the types we will discuss in Chapter 5, using the residue theorem.