Math 324

Problem 24 Solution 29 Apr 2004

(a) \[ x_2 = \frac{1}{2} \left( -x^2 - y^2 + 2z^2 \right) = \frac{1}{2} \langle 1, x^2, x^4 \rangle \]
\[ = \frac{r^2}{2} \left( -\sin^2 \theta \cos \phi - \sin^2 \theta \sin^2 \phi + 2 \cos^2 \theta \right) \]
\[ = r^2 \left( 2 \cos^2 \theta - \sin^2 \theta \right) = r^2 \left( \frac{3 \cos^2 \theta - 1}{2} \right) \]

Then \[ p^2 = \frac{3 \cos^2 \theta - 1}{2} \]

(b) \[ V = dV_2 = -x \, dx - y \, dy + 2z \, dz \]

Since \[ \epsilon_{ij} = \frac{1}{2} \left( \frac{dV_i}{dx_j} + \frac{dV_j}{dx_i} \right) \] in Cartesian coordinates,
and \[ V_i = \frac{dV_2}{dx_i} \], we have \[ \epsilon_{ij} = \frac{dV_2}{dx_i} \cdot A_{ij} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 \end{pmatrix} \]

Then the rate of energy dissipation per unit volume is \[ 2 \eta \epsilon_{ij} \epsilon_{ij} = 2 \eta \left( \frac{1^2 + 1^2 + 2^2}{2} \right) = 12 \eta \] (constant)

and in a sphere of volume \[ \frac{4}{3} \pi p^3 \], energy is dissipated at a rate \[ 12 \eta \cdot \frac{4}{3} \pi p^3 = 16 \pi \eta \pi p^3 \]

(c) According to Einstein, the normal component of stress on the sphere of radius \( p \) does work at the rate \[ 8 \pi \eta p^3 \int_0^{2 \pi} \int_0^\pi p^2 d\phi \, d\psi = 16 \pi \eta \pi p^3 \int_0^1 p^2 d\cos \phi \]

but \[ \int_0^1 p^2 d\cos \phi = \int_0^2 (3x^2 - 1)^2 dx = \frac{2}{5} \]

so more energy is dissipated than is coming in.

(The explanation is that more energy is coming in than this: the tangential components of stress also do work.)