A1 Let \( n \) be a fixed positive integer. How many ways are there to write \( n \) as a sum of positive integers, \( n = a_1 + a_2 + \cdots + a_k \), with \( k \) an arbitrary positive integer and \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1 \)? For example, with \( n = 4 \) there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

A2 Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be nonnegative real numbers. Show that
\[
\left( a_1 a_2 \cdots a_n \right)^{1/n} + \left( b_1 b_2 \cdots b_n \right)^{1/n} \\
\leq \left( (a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) \right)^{1/n}.
\]

A3 Find the minimum value of
\[
| \sin x + \cos x + \tan x + \cot x + \sec x + \csc x |
\]
for real numbers \( x \).

A4 Suppose that \( a, b, c, A, B, C \) are real numbers, \( a \neq 0 \) and \( A \neq 0 \), such that
\[
| ax^2 + bx + c | \leq | Ax^2 + Bx + C |
\]
for all real numbers \( x \). Show that
\[
|b^2 - 4ac| \leq |B^2 - 4AC|.
\]

A5 A Dyck \( n \)-path is a lattice path of \( n \) upsteps \((1, 1)\) and \( n \) downsteps \((1, -1)\) that starts at the origin \( O \) and never dips below the \( x \)-axis. A return is a maximal sequence of contiguous downsteps that terminates on the \( x \)-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.

Show that there is a one-to-one correspondence between the Dyck \( n \)-paths with no return of even length and the Dyck \((n - 1)\)-paths.

A6 For a set \( S \) of nonnegative integers, let \( r_S(n) \) denote the number of ordered pairs \((s_1, s_2)\) such that \( s_1 \in S \), \( s_2 \in S \), \( s_1 \neq s_2 \), and \( s_1 + s_2 = n \). Is it possible to partition the nonnegative integers into two sets \( A \) and \( B \) in such a way that \( r_A(n) = r_B(n) \) for all \( n \)?

B1 Do there exist polynomials \( a(x), b(x), c(y), d(y) \) such that
\[
1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y)
\]
holds identically?

B2 Let \( n \) be a positive integer. Starting with the sequence \( 1, 1/2, 1/3, \ldots, 1/n \), form a new sequence of \( n - 1 \) entries \( 3/4, 5/12, \ldots, (2n - 1)/2n(n - 1) \) by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of \( n - 2 \) entries, and continue until the final sequence produced consists of a single number \( x_n \). Show that \( x_n < 2/n \).

B3 Show that for each positive integer \( n \),
\[
n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, \ldots, |n/i|\}.
\]
(Here \( \text{lcm} \) denotes the least common multiple.)

B4 Let
\[
f(z) = az^4 + bz^3 + cz^2 + dz + e
\]
where \( a, b, c, d, e \) are integers, \( a \neq 0 \). Show that if \( r_1 + r_2 \) is a rational number and \( r_1 + r_2 \neq r_3 + r_4 \), then \( r_1 r_2 \) is a rational number.

B5 Let \( A, B, \) and \( C \) be equidistant points on the circumference of a circle of unit radius centered at \( O \), and let \( P \) be any point in the circle’s interior. Let \( a, b, c \) be the distance from \( P \) to \( A, B, C \), respectively. Show that there is a triangle with side lengths \( a, b, c \) and that the area of this triangle depends only on the distance from \( P \) to \( O \).

B6 Let \( f(x) \) be a continuous real-valued function defined on the interval \([0, 1]\). Show that
\[
\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.
\]