

The 63rd William Lowell Putnam Mathematical Competition

Saturday, December 7, 2002

A-1 Let k be a fixed positive integer. The n -th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

A-2 Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

A-3 Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \dots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

A-4 In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

A-5 Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\}.$$

A-6 Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base- b digits of n . For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

B-1 Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

B-2 Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

B-3 Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

B-4 An integer n , unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

B5 A palindrome in base b is a positive integer whose base- b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b .

B-6 Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p .)

The 62nd William Lowell Putnam Mathematical Competition

Saturday, December 1, 2001

A-1 Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

A-2 You have coins C_1, C_2, \dots, C_n . For each k , C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .

A-3 For each integer m , consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

A-4 Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R , BF bisects CG at point S , and CG bisects AE at point T . Find the area of the triangle RST .

A-5 Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

A-6 Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

B-1 Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ in the squares of an $n \times n$ grid so that the k -th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

B-2 Find all pairs of real numbers (x, y) satisfying the system of equations

$$\begin{aligned} \frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2) \\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4). \end{aligned}$$

B-3 For any positive integer n , let $\langle n \rangle$ denote the closest integer to \sqrt{n} . Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

B-4 Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \rightarrow S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

B-5 Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x . Prove that $g(x) = cx$ for some constant c .

B-6 Assume that $(a_n)_{n \geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \dots, n-1$?

The 61st William Lowell Putnam Mathematical Competition

Saturday, December 2, 2000

A-1 Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1, \dots are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

A-2 Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

A-3 The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

A-4 Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.

A-5 Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

A-6 Let $f(x)$ be a polynomial with integer coefficients. Define a sequence a_0, a_1, \dots of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

B-1 Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j , at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j , $1 \leq j \leq N$.

B-2 Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

B-3 Let $f(t) = \sum_{j=1}^N a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $\frac{d^k f}{dt^k}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \dots \text{ and } \lim_{k \rightarrow \infty} N_k = 2N.$$

[Editorial clarification: only zeroes in $[0, 1)$ should be counted.]

B-4 Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x . Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

B-5 Let S_0 be a finite set of positive integers. We define finite sets S_1, S_2, \dots of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n . Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N+a : a \in S_0\}$.

B-6 Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \dots, \pm 1)$ in n -dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

The 60th William Lowell Putnam Mathematical Competition

Saturday, December 4, 1999

A-1 Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x ,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

A-2 Let $p(x)$ be a polynomial that is nonnegative for all real x . Prove that for some k , there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^k (f_j(x))^2.$$

A-3 Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

A-4 Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

A-5 Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^1 |p(x)| dx.$$

A-6 The sequence $(a_n)_{n \geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1} a_{n-2}^2}{a_{n-2} a_{n-3}}.$$

Show that, for all n , a_n is an integer multiple of n .

B-1 Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F . Evaluate $\lim_{\theta \rightarrow 0} |EF|$.

B-2 Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

B-3 Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x, y) \rightarrow (1, 1), (x, y) \in A} (1 - xy^2)(1 - x^2y)S(x, y).$$

B-4 Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x . Suppose that $f'''(x) \leq f(x)$ for all x . Show that $f'(x) < 2f(x)$ for all x .

B-5 For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k .

B-6 Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

The 59th William Lowell Putnam Mathematical Competition

Saturday, December 5, 1998

A-1 A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A-2 Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x -axis and let B be the area of the region lying to the right of the y -axis and to the left of s . Prove that $A + B$ depends only on the arc length, and not on the position, of s .

A-3 Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

A-4 Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n .

A-5 Let \mathcal{F} be a finite collection of open discs in \mathbb{R}^2 whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection D_1, \dots, D_n in \mathcal{F} such that

$$E \subseteq \cup_{j=1}^n 3D_j.$$

Here, if D is the disc of radius r and center P , then $3D$ is the disc of radius $3r$ and center P .

A-6 Let A, B, C denote distinct points with integer coordinates in \mathbb{R}^2 . Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC .

B-1 Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

B-2 Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b) , one on the x -axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

B-3 let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C . Determine the surface area of that portion of H lying over the planar region inside P , and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.

B-4 Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

B-5 Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \dots 11.$$

Find the thousandth digit after the decimal point of \sqrt{N} .

B-6 Prove that, for any integers a, b, c , there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

The 58th William Lowell Putnam Mathematical Competition

Saturday, December 6, 1997

A-1 A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC , and F the foot of the altitude from A . What is the length of BC ?

A-2 Players $1, 2, 3, \dots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

A-3 Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx.$$

A-4 Let G be a group with identity e and $\phi : G \rightarrow G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

A-5 Let N_n denote the number of ordered n -tuples of positive integers (a_1, a_2, \dots, a_n) such that $1/a_1 + 1/a_2 + \dots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

A-6 For a positive integer n and any real number c , define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k , $1 \leq k \leq n$.

B-1 Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n , evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b .)

B-2 Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x . Prove that $|f(x)|$ is bounded.

B-3 For each positive integer n , write the sum $\sum_{m=1}^n 1/m$ in the form p_n/q_n , where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n .

B-4 Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all [integers] $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i, i} \leq 1.$$

B-5 Prove that for $n \geq 2$,

$$\underbrace{n \text{ terms}}_{2^{2^{\cdots 2}}} \equiv \underbrace{n-1 \text{ terms}}_{2^{2^{\cdots 2}}} \pmod{n}.$$

B-6 The dissection of the 3-4-5 triangle shown below (into four congruent right triangles similar to the original) has diameter $5/2$. Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)