Claim: $\dfrac{d}{dx}\cos x = -\sin x$.

Proof: By the definition of the derivative, we have

$$\frac{d}{dx}\cos x \quad = \quad \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}.$$

Using the addition formula for cosine and rearranging the terms somewhat, we get

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{(\cos(h)-1)\cos(x)-\sin(h)\sin(x)}{h} \\
&= \lim_{h\to 0}\left[\left(\frac{\cos(h)-1}{h}\right)\cos(x)-\left(\frac{\sin(h)}{h}\right)\sin(x)\right].
\end{aligned}$$

Now we use the fact that the limit of the difference is the difference of the limits to write

$$\frac{d}{dx}\cos x \quad = \quad \lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)\cos(x)-\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)\sin(x). \tag{1}$$

Now $\cos(x)$ and $\sin(x)$ do not involve h, so they are constant as far as the limiting process is concerned. Furthermore, we know from arguments given in class that

$$\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) \quad = \quad 1$$

and

$$\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) \quad = \quad 0.$$

Plugging these limits into equation (1), we get

$$\begin{aligned}
\frac{d}{dx}\cos x &= 0\cdot\cos(x)-1\cdot\sin(x) \\
&= -\sin(x),
\end{aligned}$$

as required.