1. A spring hangs from a hook on the ceiling. When a 5-lb weight is attached to the spring, the spring stretches to a length of 27 inches. When a 10-lb weight is attached to the spring, it stretches to a length of 33 inches.

   (a) Assuming a linear model is correct, find L, the length of the spring, as a function of w, the amount of weight attached to it.

   Solution: Assume $L(w) = mw + b$ for some numbers m and b. Given that $L(5) = 27$ and $L(10) = 33$, we find first that

   $$m = \frac{33 - 27}{10 - 5} = \frac{6}{5}$$

   Then since $27 = \frac{6}{5} \cdot 5 + b$, we find that $b = 21$. The function is

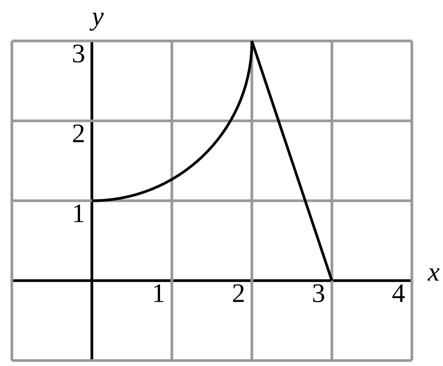
   $$L(w) \quad = \quad \frac{6}{5}w + 21.$$

   (b) What is the L-intercept in this model? What is the meaning of the L-intercept in terms of the spring?

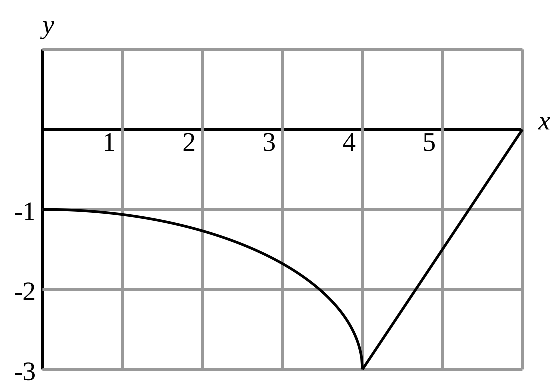
   Solution: We have $L(0) = 21$ inches. This is the natural length of the spring. That is, the length of the spring when no weight is attached to it.

2. The figure below shows the graph of a function f. Sketch a graph of the function g, where g is given by $g(x) = -f(x/2)$.

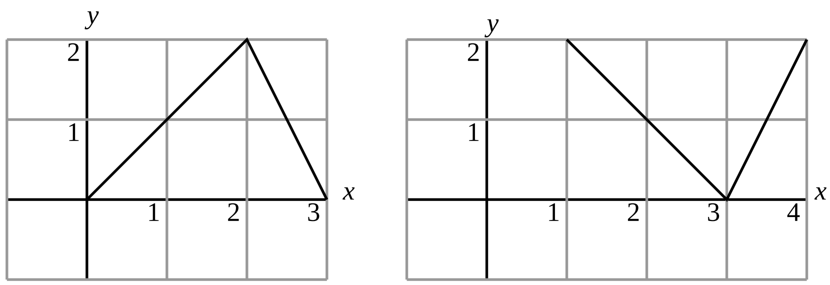
   (Two grids are provided; one is for practice. Be sure to indicate which grid contains your final answer, and don't forget to label the x and y axes and include tickmark labels.)



Solution:



3. The diagram at the left shows the graph of a function f. The diagram at the right shows the graph of a function g given by $g(x) = A + Bf(x + C)$. Find the values of A, B, and C.



Solution: From the left-hand diagram to the right-hand diagram, the graph has been reflected across the x-axis, translated upward two units, and translated to the right one unit. The values are: $A = 2$, $B = -1$, and $C = -1$.

4. Find the indicated limits.

(a) $\lim\limits_{x\to 5} \dfrac{x^2 - x - 20}{x - 5}$.

Solution: The given function is the same as $\dfrac{(x-5)(x+4)}{x-5}$, which is undefined at $x = 5$, but elsewhere is equal to $x+4$. Its limit as $x \to 5$ must therefore be the same as $\lim\limits_{x\to 5} x+4$, which is 9.

(b) $\lim\limits_{x\to 5^+} \dfrac{\sqrt{x+4} - 4}{x - 5}$.

Solution: The expression has a zero denominator (and a non-zero numerator) at 5, so we suspect a limit of $\pm\infty$. We investigate

$$\frac{\sqrt{5^+ + 4} - 4}{(5^+ - 5)} \quad \approx \quad \frac{3 - 4}{(0^+)}.$$

This is a negative number divided by a very tiny positive number, so the result is a large negative number. We conclude that

$$\lim_{x\to 5^+} \frac{\sqrt{x+4} - 4}{x - 5} \quad = \quad -\infty.$$

(c) $\lim_{x\to 5} \frac{\sqrt{x+4}-4}{x-4}$

Solution: When we plug in $x = 5$, we get

$$\frac{\sqrt{9}-4}{5-4} \quad = \quad \frac{-1}{1}$$

which is -1. Since nothing goes wrong, we conclude that

$$\lim_{x\to 5} \frac{\sqrt{x+4}-4}{x-4} \quad = \quad -1.$$

(d) $\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5}$.

Solution: We multiply top and bottom by $\sqrt{x+4}+3$ to get

$$\frac{x+4-9}{(x-5)(\sqrt{x+4}+3)} \quad = \quad \frac{x-5}{(x-5)(\sqrt{x+4}+3)},$$

which is undefined at $x = 5$, but everywhere else is equal to $\frac{1}{\sqrt{x+4}+3}$. The limit as $x \to 5$ must therefore be the same as $\lim_{x\to 5} \frac{1}{\sqrt{x+4}+3}$, which is $\frac{1}{6}$.

5. Let $f(x) = x^2 - 3x$.

(a) Find an equation for the secant line that intersects the curve $y = f(x)$ at the point where $x = 2$ and the point where $x = 3$.

Solution: The points in question are $(2, f(2)) = (2, -2)$ and $(3, f(3)) = (3, 0)$. The slope of the secant line is thus

$$\frac{0 - (-2)}{3 - 2} \quad = \quad 2.$$

Since we know the line passes through $(3, 0)$, we can use the point-slope form of the equation for a line to get

$$y \quad = \quad 2(x - 3).$$

(b) Let $m(h)$ denote the slope of the secant line through the point $(2, f(2))$ and the point $(2 + h, f(2 + h))$. Find a formula for $m(h)$. Simplify as far as possible.

Solution: The slope is given by

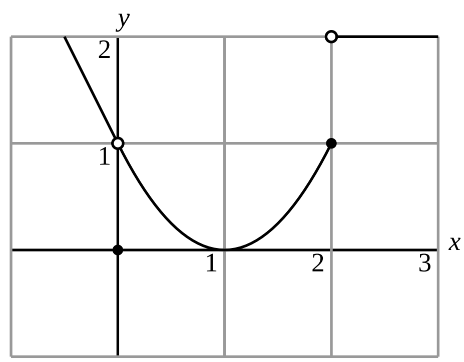
$$\begin{aligned} m(h) &= \frac{f(2+h) - f(2)}{(2+h) - 2} \\ &= \frac{(2+h)^2 - 3(2+h) - (-2)}{h} \\ &= \frac{(4 + 4h + h^2) - (6 + 3h) + 2}{h} \\ &= \frac{h^2 + h}{h} \\ &= \begin{cases} h + 1 & \text{if } h \neq 0 \\ \text{undefined} & \text{if } h = 0. \end{cases} \end{aligned}$$

6. Let f be the function given by

$$f(x) = \begin{cases} 1-2x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ (x-1)^2 & \text{if } 0<x\leq 2 \\ 2 & \text{if } x>2 \end{cases}$$

At each point, indicate whether f is continuous, has a removable discontinuity, has a jump discontinuity, or has an infinite discontinuity. *Justify your answers.*

Solution: A graph of f will be helpful here. We have



(a) $x = 0$

Solution: We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1-2x = 1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-1)^2 = 1$, so $\lim_{x\to 0} f(x) = 1$. However, $f(0) = 0$. Since $\lim_{x\to 0} f(x) \neq f(0)$, we conclude that f has a removable discontinuity at $x = 0$.

(b) $x = 2$

Solution: We have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-1)^2 = 1$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2 = 2$. Since $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ are both finite, but are unequal, we conclude that f has a jump discontinuity at $x = 2$.

7. Use the Intermediate Value Theorem to show that the equation

$$x^4 - 5x^3 + 2 = 0$$

has at least one solution.

Solution: Let $f(x) = x^4 - 5x^3 + 2$. Since f is a polynomial, it is continuous everywhere. In particular, f is continuous on the interval $[0, 1]$. Furthermore, we find that

$$f(0) = 2 \quad \text{and} \quad f(1) = -2.$$

Since the number 0 lies between -2 and 2, by the Intermediate Value Theorem, we know that there is a number c in the interval $(0, 1)$ such that $f(c) = 0$. That is, c is a solution to the equation

$$x^4 - 5x^3 + 2 = 0.$$

So at least one such solution exists.