Math 101(04) Hour Exam II – Solution 22 October 2001

1. Find $f'(x)$. Do not simplify.

   (a) $f(x) = \dfrac{x^2+1}{3x^3-5}$.
   Solution: By the quotient rule, we get

$$f'(x) = \frac{(3x^3-5)\cdot(2x)-(x^2+1)\cdot(9x^2)}{(3x^3-5)^2}.$$

   (b) $f(x) = \cos(\sqrt{1-x^2})$
   Solution: We get

$$f'(x) = -\sin(\sqrt{1-x^2})\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x).$$

2. Suppose F, f, and g are functions and that $F = f \circ g$. Use the information given in the following table of values to find an equation for the line tangent to the curve $y = F(x)$ at $x = 1$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	1	3	3	2
2	3	-1	4	-2
3	2	-2	-1	0

Solution: First, we have $F(1) = f(g(1)) = f(3) = 2$, so our line passes through the point $(1, 2)$. Next, to find the slope, we use the fact that

$$\begin{aligned} F'(1) &= f'(g(1)) \cdot g'(1) \\ &= f'(3) \cdot 2 \\ &= (-2) \cdot 2 \\ &= -4. \end{aligned}$$

Using the point-slope form for the equation of a line, we get

$$y - 2 = -4(x - 1).$$

3. Find an equation for the line tangent to the curve $y = \sin(2x)$ at the point where $x = \dfrac{\pi}{3}$. Leave your answer in exact form (in terms of π and square roots).

Solution: We have

$$\begin{aligned} y\left(\frac{\pi}{6}\right) &= \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2}. \end{aligned}$$

To find the slope, we take

$$y' = 2\cos(2x)$$

and evaluate at $x = \dfrac{\pi}{3}$. We get

$$\begin{aligned} y'\left(\frac{\pi}{3}\right) &= 2\cos\left(\frac{2\pi}{3}\right) \\ &= -1. \end{aligned}$$

The equation for the line is

$$\left(y - \frac{\sqrt{3}}{2}\right) = -\left(x - \frac{\pi}{3}\right).$$

4. Find the slope of the line tangent to the curve $x^3 + 2y^3 - x^2y^4 = x$ at the point $(2, -1)$.

Solution: The slope is $\dfrac{dy}{dx}$, which we find by implicit differentiation. We get

$$3x^2 + 6y^2y' - 4x^2y^3y' - 2xy^4 \quad = \quad 1,$$

which we solve to get

$$y' \quad = \quad \frac{1 + 2xy^4 - 3x^2}{6y^2 - 4x^2y^3}.$$

At the point $x = 2$, $y = -1$, we have

$$\begin{aligned} y' &= \frac{1 + 4 - 12}{6 + 16} \\ &= -\frac{7}{22}. \end{aligned}$$

5. A particle moves along a straight line with position x (in inches) as a function of t (in seconds) given by

$$x(t) = t^3 - 6t^2 + 9t.$$

(a) Give the particle's direction of motion and its velocity at time $t = 2$. Be sure to include the proper units.

Solution: We have

$$x'(t) = 3t^2 - 12t + 9.$$

The velocity at $t = 2$ is $x'(2) = 3 \times 4 - 12 \times 2 + 9 = -3$. The particle is moving in the negative direction at 3 inches per second.

(b) At what times does the particle change direction?

Solution: The particle changes direction when its velocity changes sign. Since the velocity is a continuous function, this can happen only when the velocity is zero. We have

$$\begin{aligned} x'(t) &= 3t^2 - 12t + 9 \\ &= 3(t^2 - 4t + 3) \\ &= 3(t-1)(t-3), \end{aligned}$$

which changes sign when $t = 1$ and when $t = 3$.

6. The marketing department at Amalgamated Widget finds that the cost C (in dollars) of producing x cartons of squash balls given by

$$C(x) = 3000 + 12x - 0.01x^2.$$

Find and interpret the marginal cost at a production level of 280 cartons of squash balls.

Solution: We have

$$C'(x) = 12 - 0.02x$$

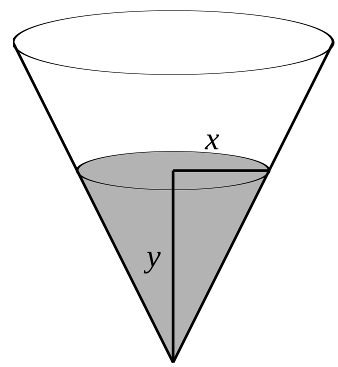
so that the marginal cost at $x = 280$ is given by

$$C'(280) = 12 - 0.02 \times 280 = 6.40.$$

When the production level is 280 cartons of squash balls, the additional cost of producing another carton is approximately \$6.40.

7. Concrete is being poured into a conical tank (vertex down) at the rate of 2 cubic feet per second. The tank is 10 feet deep and has a radius at the top of 5 feet. At what rate is the level of concrete in the tank rising when it has been filled to a depth of 6 feet? (The volume of a cone with radius r and height h is given by $\frac{1}{3}\pi r^2 h$).

Solution: As in the drawing, let x and y denote the depth and radius of the already-poured concrete. By similar triangles, we find that $x = \frac{y}{2}$, so we can express the volume V of the already-poured concrete as



$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y \\ &= \frac{\pi}{12}y^3. \end{aligned}$$

Differentiating both sides with respect to t, we find that

$$\begin{aligned} \frac{dV}{dt} &= \frac{3\pi}{12}y^2\frac{dy}{dt} \\ &= \frac{\pi y^2}{4}\frac{dy}{dt}. \end{aligned}$$

When $y = 6$, the equation becomes

$$\frac{dV}{dt} = 9\pi\frac{dy}{dt}.$$

We are given that $\frac{dV}{dt} = 2$, so we can solve for $\frac{dy}{dt}$ (which is what we want). We get

$$\frac{dy}{dt} = \frac{2}{9\pi} \text{ feet per second.}$$