**Math 101(04)** **Hour Exam III – Solution** **13 November 2001**

1. Find the local linearization $L(x)$ for the function $f(x) = \dfrac{1}{1+x^2}$ at $a = 3$.

   Solution: We have

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

   so that

$$\begin{aligned} f'(3) &= -\frac{6}{10^2} \\ &= -\frac{6}{100}. \end{aligned}$$

   We also compute

$$f(3) = \frac{1}{10},$$

   and conclude that

$$L(x) = \frac{1}{10} - \frac{6}{100}(x-3).$$

2. The Whizzo Confectionery Company is planning to market a new candy called the Dental Delight. It consists of a (spherical) steel ball with radius 5 mm, covered with a milk chocolate coating $\frac{1}{4}$ mm thick.

   Use differentials to approximate the volume of milk chocolate in each Dental Delight. Be sure to include units in your answer.

   (The volume $V$ of a sphere of radius $r$ is given by $V = \frac{4}{3}\pi r^3$.)

   Solution: We have

$$dV = 4\pi r^2,$$

   so when $r = 5$ mm, we get

$$\begin{aligned} dV &= 4\pi(25)\ \text{mm}^2\, dr \\ &= 100\pi \times \frac{1}{4}\ \text{mm}^3 \\ &= 25\pi\ \text{mm}^3. \end{aligned}$$

3. In a computer model of bacterial growth, the function $P(t)$ represents the population of a certain bacterium at time $t$. It is known that $P(10) = 5000$ and that, in general, the rate of growth in the population follows the rule

$$P'(t) = 0.02P(t).$$

Use a linear approximation to estimate $P(10.5)$.

Solution: We know $P(10) = 5000$, and we can find $P'(10)$, because we know that

$$\begin{aligned} P'(10) &= 0.02P(10) \\ &= 0.02(5000) \\ &= 100. \end{aligned}$$

The linear approximation $L(t)$ is thus

$$L(t) = 5000 + 100(t - 10),$$

and so we get

$$\begin{aligned} P(10.5) &\approx 5000 + 100(10.5 - 10) \\ &= 5050. \end{aligned}$$

At $t = 10.5$, we expect the population will be approximately 5050.

4. Let $f(x) = x + 2\sin x$. Find the absolute maximum and minimum values of $f$ on the interval $[0, \pi]$.

Solution: The given function is continuous everywhere, so it's continuous on the given interval. Thus the maximum and minimum values must occur at critical numbers or at the endpoints of the interval. To find the critical numbers, we compute

$$f'(x) = 1 + 2\cos x.$$

Since $f'(x)$ is defined for all values of $x$, the only critical numbers will be those where $f'(x) = 0$. We solve

$$\begin{aligned} 1 + 2\cos x &= 0 \\ \cos x &= -\frac{1}{2}. \end{aligned}$$

The only value of $x$ in $[0, \pi]$ for which $\cos x = -\frac{1}{2}$ is $x = \frac{2\pi}{3}$. Evaluating $f$ at this critical number and at the endpoints of the interval, we get

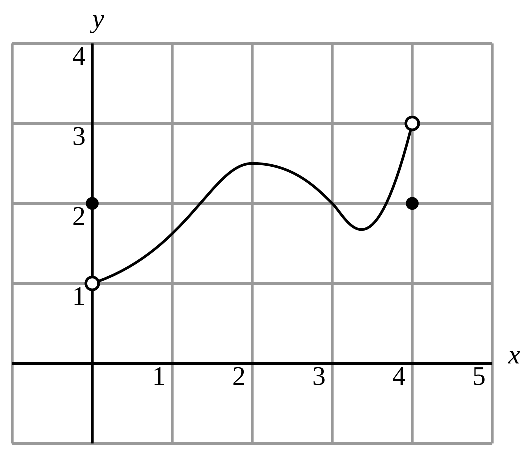
$$\begin{aligned} f(0) &= 0 \\ f\left(\frac{2\pi}{3}\right) &= \frac{2\pi}{3} + \sqrt{3} \\ f(\pi) &= \pi. \end{aligned}$$

The largest of these values is $\frac{2\pi}{3} + \sqrt{3}$ and the smallest is 0. Thus the absolute maximum and minimum values of $f$ in the given interval are, respectively, $\frac{2\pi}{3} + \sqrt{3}$ and 0.

5. Sketch the graph of a function $f$ satisfying the following:

- $f$ is defined on $[0, 4]$, and continuous on $(0, 4)$.
- $f$ is differentiable on $(0, 4)$.
- $f$ has a local maximum at 2
- $f'(3) < 0$
- $f$ has no absolute maximum or minimum on $[0, 4]$.

Here is one solution:



6. Let $f(x) = x^3 + 4x + 6$. Prove that $f$ has at most one root.

Solution: Suppose not. That is, suppose there are numbers $a$ and $b$ (with $a < b$, say) such that $f(a) = 0$ and $f(b) = 0$. Since $f$ is a polyonomial, it is continuous and differentiable on any interval. In particular, $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Thus by the Mean Value Theorem, there is a number $c$ in $(a, b)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{0 - 0}{b - a} \\ &= 0. \end{aligned}$$

Now $f'(x) = 3x^2 + 4$, so if $f'(c) = 0$, we must have

$$3c^2 + 4 = 0,$$

which implies that $3c^2 = -4$, and thus that $c^2 = -\frac{4}{3}$. But this cannot happen, since the square of any real number must be non-negative. Thus we have a contradiction, and our assumption that $f$ has two distinct roots must be false. Therefore $f$ has at most one root.