Math 101(04) **Hour Exam IV – Solution** **30 November 2001**

1. Let f be the function given by $f(x) = x^4 - ax^2$, where a is a positive constant.

   (a) List the intervals on which f is increasing and the intervals on which f is decreasing. (Your answer will be in terms of a.)

   Solution: We have $f'(x) = 4x^3 - 2ax = 2x(2x^2 - a)$, from which we get that the critical numbers for f are 0 and $\pm\sqrt{\frac{a}{2}}$. Here is a sign chart for $f'(x)$:

	$(-\infty, -\sqrt{\frac{a}{2}})$	$(-\sqrt{\frac{a}{2}}, 0)$	$(0, \sqrt{\frac{a}{2}})$	$(\sqrt{\frac{a}{2}}, \infty)$
$2x$:	$-$	$-$	$+$	$+$
$2x^2 - a$:	$+$	$-$	$-$	$+$
$f'(x)$:	$-$	$+$	$-$	$+$

   The function f is increasing on the intervals $\left[-\sqrt{\frac{a}{2}}, 0\right]$ and $\left[\sqrt{\frac{a}{2}}, \infty\right)$, and decreasing on the intervals $\left(-\infty, -\sqrt{\frac{a}{2}}\right]$ and $\left[0, \sqrt{\frac{a}{2}}\right]$.

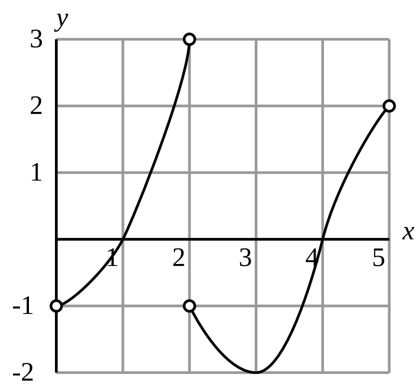
   (b) List the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. (Your answer will be in terms of a.)

   Solution: We have $f''(x) = 12x^2 - 2a = 2(6x^2 - a)$, so the second derivative of f can change sign only at $x = \pm\sqrt{\frac{a}{6}}$. Here is a sign chart for $f''(x)$:

	$(-\infty, -\sqrt{\frac{a}{6}})$	$(-\sqrt{\frac{a}{6}}, \sqrt{\frac{a}{6}})$	$(\sqrt{\frac{a}{6}}, \infty)$
$6x^2 - a$:	$+$	$-$	$+$

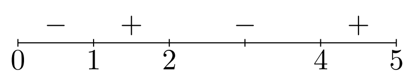
   The graph of f is concave upward on the intervals $\left(-\infty, -\sqrt{\frac{a}{6}}\right)$ and $\left(\sqrt{\frac{1}{6}}, \infty\right)$ and concave downward on the interval $\left(-\sqrt{\frac{a}{6}}, \sqrt{\frac{a}{6}}\right)$.

2. The function f is continuous on $[0,5]$ and differentiable at all points of $(0,5)$ except $x=2$. Here is a graph of the *derivative* of f. That is, this graph shows the curve $y=f'(x)$.



(a) At what values of x does f have a local maximum? At what values of f does x have a local minimum?

Solution: From the graph of f', we can see that the critical numbers of f are 1, 2, and 4. Here is a sign chart for f':



The function f has a local minimum at $x=1$, a local maximum at $x=2$, and a local minimum at $x=4$. (There is also an endpoint maximum at $x=0$ and an endpoint maximum at $x=5$.)

(b) On what intervals is the graph of f concave upward? On what intervals is the graph of f concave downward?

Solution: We know that the graph of f is concave upward on intervals where f' is increasing, and concave downward on intervals where f' is decreasing. From the picture, we see that f' is increasing, and therefore the graph of f is concave upward, on the intervals

$$(0,2) \quad \text{and} \quad (3,5).$$

Again from the picture, f' is decreasing, and therefore the graph of f is concave downward, on the interval $(2,3)$.

3. (a) Find $\displaystyle\lim_{x\to-\infty}\left(\frac{2x^3+4}{|x|(3x-1)^2}\right)$.

Solution: We have

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{2x^3+4}{(-x)(9x^2-6x+1)}\right) &= -\lim_{x\to\infty}\frac{2x^3+4}{9x^3-6x^2+x}\\
&= -\lim_{x\to\infty}\frac{2+\frac{4}{x^3}}{9-\frac{6}{x}+\frac{1}{x^2}}\\
&= -\frac{2}{9}.
\end{aligned}$$

(b) Find $\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x^2+x}-x}$.

Solution: The form is $\dfrac{1}{\infty-\infty}$, so we need to do some manipulation. We multiply top and bottom by $\sqrt{x^2+x}+x$ to get

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{x^2+x}+x}{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)} &= \lim_{x\to\infty}\frac{\sqrt{x^2+x}+x}{x^2+x-x^2}\\
&= \lim_{x\to\infty}\frac{\sqrt{x^2+x}+x}{x}\\
&= \lim_{x\to\infty}\left(\sqrt{\frac{x^2+x}{x^2}}+1\right)\\
&= \lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x}}+1\right)\\
&= 2.
\end{aligned}$$

4. The equation $x^3 - 30 = 0$ has a single root, $x = \sqrt[3]{30}$.

(a) Set up the iteration rule you would use in applying Newton's method to find the root of the equation $x^3 - 30 = 0$. Your rule should give a formula for x_{n+1} in terms of x_n.

Solution:

$$x_{n+1} = \boxed{x_n - \frac{x_n^3 - 30}{3x_n^2}}$$

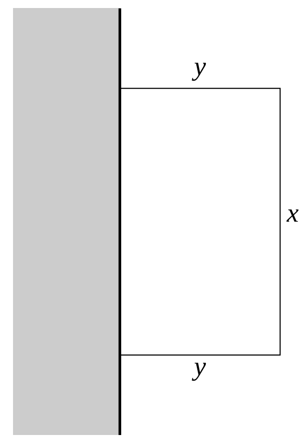
(b) Using the initial guess $x_0 = 3$, find x_1. Leave your answer in exact form.

Solution: We have

$$\begin{aligned} x_1 &= 3 - \frac{(3)^3 - 30}{3(3)^2} \\ &= 3 - \frac{-3}{27} \\ &= 3 + \frac{1}{9} \\ &= \frac{28}{9}. \end{aligned}$$

5. A rectangular garden with an area of 1000 square meters is to be laid out beside a straight river. The bank of the river provides one side of the garden; along the other three sides we need to put up fencing. Find the minimum amount of fencing needed to enclose the garden.

Solution:
Let x denote the length of the side of the garden parallel to the river and let y denote the length of the adjacent side. Let P denote the amount of fencing we will need. Then we have



$$P = x + 2y,$$

and our goal is to minimize P. We also know that the area of the garden, xy, is equal to 1000, so we have

$$\begin{aligned} xy &= 1000 \\ y &= \frac{1000}{x}. \end{aligned}$$

Subsituting for y in the expression for P, we get

$$P = x + \frac{2000}{x}.$$

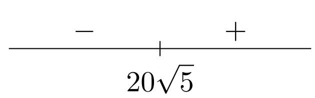
We find that

$$P' = 1 - \frac{2000}{x^2}.$$

so the critical number of P (with $x > 0$) satisfies

$$1 = \frac{2000}{x^2}.$$

Thus $x = \sqrt{2000} = 20\sqrt{5}$ is a critical number for P. Here is a sign chart for P':



We have an absolute minimum at $x = 20\sqrt{5}$. Thus for the minimum amount of fencing, we take

$$x = 20\sqrt{5} \quad \text{and} \quad y = \frac{1000}{x} = \frac{50}{\sqrt{5}} = 10\sqrt{5}.$$

The minimum amount of fencing is given by

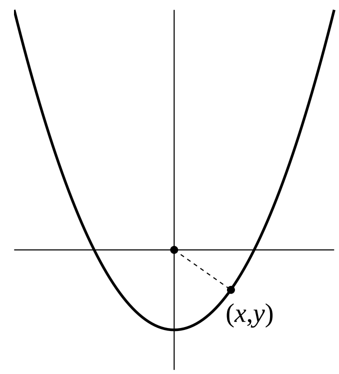
$$x + 2y \quad = \quad 40\sqrt{5} \text{ meters.}$$

6. Find the points on the curve $y = x^2 - 1$ that are closest to the origin.

Solution:

Let (x, y) be a point on the curve. We want to minimize the distance from (x, y) to the origin. We may as well minimize the square of the distance to the origin, that is

$$D = x^2 + y^2.$$



We know that $y = x^2 - 1$, so we can write

$$D = x^2 + (x^2 - 1)^2.$$

Taking a derivative, we get

$$\begin{aligned} D' &= 2x + 4x(x^2 - 1) \\ &= 2x(1 + 2(x^2 - 1)) \\ &= 2x(2x^2 - 1). \end{aligned}$$

The critical numbers are $x = 0$ and $x = \pm\sqrt{\frac{1}{2}}$. Here is a sign chart for D':

		$-\sqrt{\frac{1}{2}}$		0		$\sqrt{\frac{1}{2}}$	
$2x$:	$-$		$-$		$+$		$+$
$2x^2 - 1$:	$+$		$-$		$-$		$+$
$D'(x)$:	$-$		$+$		$-$		$+$

There is a local maximum at $x = 0$ and there are local (and absolute) minima at $x = \pm\sqrt{\frac{1}{2}}$. The points on the curve that are closest to the origin are

$$\left(\pm\sqrt{\frac{1}{2}}, -\frac{1}{2}\right)$$