Section 4.2

These problems are proofs, so the solutions must be written in complete sentences. Here are sample solutions to the problems that were assigned.

17. *Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.*

    Solution: We first use the Intermediate Value Theorem to show that the equation has at least one root. Let $f(x) = x^5 + 10x + 3$. Then

$$f(-1) = (-1)^5 + 10(-1) + 3 = -8$$

and

$$f(0) = 0^5 + 10(0) + 3 + 3.$$

Since f is a polynomial, we know it is continuous on every interval. In particular, f is continuous on $[-1, 0]$, and since 0 is a number between $f(-1)$ and $f(0)$, we know by the Intermediate Value Theorem that there exists a number c in $(-1, 0)$ such that $f(c) = 0$. That is, c is a root of the equation $x^5 + 10x + 3 = 0$. So the given equation has at least one root.

To prove that the equation has at most one root, we assume that there are two distinct roots and draw a contradiction. So let us assume that there are distinct numbers a and b such that $f(a) = 0$ and $f(b) = 0$. Then since f is a polynomial, we know that f is continuous and differentiable on every interval, and in particular that f is continuous on $[a, b]$ and differentiable on (a, b). Thus by the Mean Value Theorem there is a number c in (a, b) such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{0 - 0}{b - a} \\ &= 0. \end{aligned}$$

However, we also know that

$$f'(x) = 5x^4 + 10$$

and since $x^4 \geq 0$ for all real values of x, it follows that $f'(x) \geq 10$ for all real values of x. This contradicts the existence of a number c with $f'(c) = 0$. Since our original assumption (that the equation has two distinct roots) led to a contradiction, that assumption must be false. The equation has at most one root.

19. *Show that the equation* $x^5 - 6x + c = 0$ *has at most one root in the interval* $[-1, 1]$.

Solution: To show that this equation has at most one root in the given interval, we will assume that it has two distinct roots in the interval, and draw a contradiction.

To begin, let $f(x) = x^5 - 6x + c$. Assume that there are distinct numbers a and b in the interval $[-1, 1]$ such that $f(a) = 0$ and $f(b) = 0$.

Since f is a polynomial, we know that f is continuous and differentiable on every interval. In particular, f is continuous on $[a, b]$ and differentiable on (a, b). Thus by the Mean Value Theorem there is a number q in (a, b) such that

$$\begin{aligned} f'(q) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{0 - 0}{b - a} \\ &= 0. \end{aligned}$$

Furthermore, since a and b lie in the interval $[-1, 1]$ and q lies between a and b, we know that q lies in $[-1, 1]$ as well. That is, there exists a number q with $|q| < 1$ and $f'(q) = 0$.

Meanwhile, we have

$$f'(x) = 5x^4 - 6.$$

Now since $|q| < 1$, we know that $q^4 < 1$. Multiplying through by 5, we get $5q^4 < 5$ and subtracting 6 from each side, we get

$$5q^4 - 6 < 5 - 6 = -1.$$

But the expression on the left is exactly $f'(q)$, and so we have shown that $f'(q) < -1$. Since we also know that $f'(q) = 0$, we have reached

a contradiction. (Zero is *not* less than -1.) Thus our assumption that there were two roots of the given equation in the interval $[-1, 1]$ must be false, and we conclude that the given equation has at most one root in the given interval.

32. *At* 2:00 PM *a car's speedometer reads* 30 mi/h. *At* 2:10 PM *it reads* 50 mi/h. *Show that at some time between* 2:00 *and* 2:10 *the acceleration is exactly* 120 mi/h^2.

    Solution: Let $s(t)$ denote the speed of the car (in miles per hour) at time t (in hours after 2:00 PM). We are given that $s(0) = 30$ and $s(1/6) = 50$. Since $s(t)$ represents a physical quantity, we may assume that it is continuous and differentiable on every time interval. In particular, s is continuous on $[0, 1/6]$ and differentiable on $(0, 1/6)$. Thus by the Mean Value Theorem, we know that there is a number c in the interval $(0, 1/6)$ such that

$$\begin{aligned} s'(c) &= \frac{s(1/6) - s(0)}{1/6 - 0} \\ &= \frac{50 - 30}{1/6} \\ &= 120 \text{ mi/h}^2. \end{aligned}$$