1. A photograph of a certain tree taken on January 1, 1980 shows that the tree was 12 ft tall on that date. On January 1, 1995, another photograph shows that the same tree was 37 ft tall.

   (a) Assuming a linear model is correct, find D, the current date (measured in years CE), as a function of h, the height of the tree.

   Solution: Assume $D(h) = mh+b$ for some numbers m and b. Given that $D(12) = 1980$ and $D(37) = 1995$, we find first that

   $$m = \frac{1995 - 1980}{37 - 12} = \frac{15}{25} = \frac{3}{5}.$$

   Then since $1980 = \frac{3}{5} \cdot 12 + b$, we find that $b = \frac{9864}{5}$. The function is

   $$D(h) = \frac{3}{5}h + \frac{9864}{5}.$$

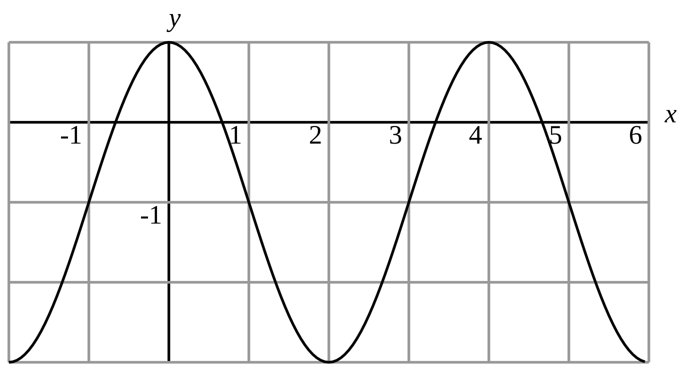
   (b) What is the slope in this model? What is the meaning of the slope in terms of the tree?

   Solution: The slope is 3/5 years per foot. The tree takes 3/5 of a year to grow one foot.

   (c) What is the D-intercept in this model? What is the meaning of the D-intercept in terms of the tree?

   Solution: We have $D(0) = \frac{9854}{5} = 1972.8$. This is the date when the tree's height was zero. If the linear model is correct, then the tree started growing sometime in September 1972.

2. The diagram shows the graph of a function f given by $f(x) = A\sin(B(x + C)) + D$. Find possible values for A, B, C, and D.



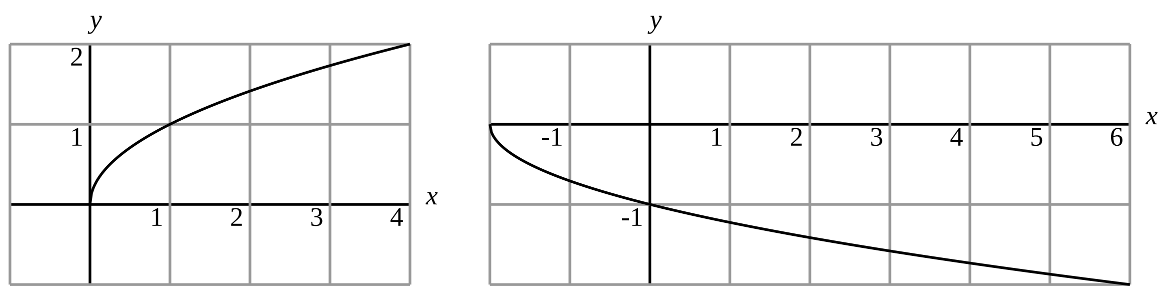
Solution: The sine wave in the picture is centered (vertically) at $y = -1$, so we may take $D = -1$. The maximum values of the pictured wave are at $y = 1$, which is two units higher than $y = -1$, so we may take $A = 2$. The wave in the picture begins at $x = -1$, so we may take $c = 1$. The function $\sin(x)$ has a period of 2π, and the given wave has a period of 4. So the given wave has been compressed horizontally by a factor of $2\pi/4 = \pi/2$. We take $B = \pi/2$. The function f is given by

$$f(x) \quad = \quad 2\sin\left(\frac{\pi}{2}(x+1)\right) - 1.$$

3. The graph on the left shows the curve $y = \sqrt{x}$. In the grid on the right, plot the graph of a function f given by

$$f(x) \quad = \quad -\sqrt{\frac{1}{2}(x+2)}.$$

Explain how to do this without using a graphing calculator or playing "connect-the-dots."



Solution: The graph of f is obtained from the original square-root graph by

(a) reflecting across the x-axis,

(b) stretching horizontally by a factor of 2, and

(c) translating two units to the left.

The resulting graph is as shown.

4. Let $f(x) = x^2 - 2x$ and $g(x) = \sqrt{3x+2}$.

(a) Find a formula for $f \circ g(x)$.
Solution: We have

$$\begin{aligned} f(g(x)) &= f(\sqrt{3x+2}) \\ &= (\sqrt{3x+2})^2 - 2\sqrt{3x+2} \\ &= 3x + 2 - 2\sqrt{3x+2}. \end{aligned}$$

(b) Find a formula for $f \circ f(x)$. Simplify as far as possible.

Solution: We have

$$\begin{aligned} f(f(x)) &= f(x^2-2x) \\ &= (x^2-2x)^2 - 2(x^2-2x) \\ &= x^4 - 4x^3 + 4x^2 - 2x^2 + 4x \\ &= x^4 - 4x^3 + 2x^2 + 4x. \end{aligned}$$

5. Find the indicated limits.

(a) $\lim\limits_{x \to 2} \dfrac{x^2+x-6}{x-2}$.

Solution: The given function is the same as $\dfrac{(x-2)(x+3)}{x-2}$, which is undefined at $x=2$, but elsewhere is equal to $x+3$. Its limit as $x \to 2$ must therefore be the same as $\lim\limits_{x\to 2} x+3$, which is 5.

(b) $\lim\limits_{x \to 4} \dfrac{\sqrt{x}-2}{x-4}$.

Solution: We multiply top and bottom by $\sqrt{x}+2$ to get

$$\frac{x-4}{(x-4)(\sqrt{x}+2)},$$

which is undefined at $x=4$, but everywhere else is equal to $\dfrac{1}{\sqrt{x}+2}$. The limit as $x \to 4$ must therefore be the same as $\lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}+2}$, which is $\dfrac{1}{4}$.

(c) $\lim\limits_{x \to 3^-} \dfrac{x-5}{(x-3)(x-1)}$.

Solution: The expression has a zero denominator at 3, so we suspect a limit of $\pm\infty$. We investigate

$$\frac{3^- - 5}{(3^- - 3)(3^- - 1)} \approx \frac{-2}{(0^-)(2)}.$$

This is a negative number divided by a very tiny negative number, so the result is a large positive number. We conclude that

$$\lim_{x\to 3^-} \frac{x-5}{(x-3)(x-1)} = \infty.$$

(d) $\lim_{x \to 2} x^2 + 5x + 1$.

Solution: The given function is a polynomial, so it's continuous for all values of x. We get

$$\begin{aligned}\lim_{x \to 2} x^2 + 5x + 1 &= (2)^2 + 5 \cdot 2 + 1 \\ &= 15.\end{aligned}$$

6. Let $f(x) = 4x - x^2$.

(a) Let $m(a)$ denote the slope of the secant line through the points $(2, f(2))$ and $(a, f(a))$. Find a formula for $m(a)$. Simplify as far as possible.

Solution: We have

$$\begin{aligned} m(a) &= \frac{f(a) - f(2)}{a-2} \\ &= \frac{4a - a^2 - 4}{a-2} \\ &= \frac{-(a^2 - 4a + 4)}{a-2} \\ &= \frac{-(a-2)^2}{a-2} \\ &= \begin{cases} -(a-2) & \text{if } a \neq 2 \\ \text{undefined} & \text{if } a = 2. \end{cases} \end{aligned}$$

(b) Find a value of a so that the slope of the secant line through $(2, f(2))$ and $(a, f(a))$ is 5.

Solution: We simply solve $m(a) = 5$. We get

$$\begin{aligned} 5 &= m(a) \\ &= -(a-2) \\ &= -a + 2 \end{aligned}$$

from which we get $a = -3$.

7. Let f be the function given by

$$f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 1 - (x-1)^2 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ \sin(x-2) & \text{if } x > 2 \end{cases}$$

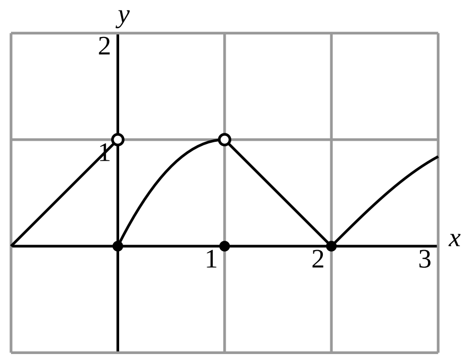
At each point, indicate whether f is continuous, has a removable discontinuity, has a jump discontinuity, or has an infinite discontinuity. Give reasons.

(a) $x = 0$

(b) $x = 1$

(c) $x = 2$

Solution: Here's a sketch of the function.



(a) We have $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x + 1 = 1$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 - (x-1)^2 = 0$. Since the left- and right-hand limits at $x = 0$ are both finite, but are unequal, the function f has a jump discontinuity at $x = 0$.

(b) We have $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 - (x-1)^2 = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x) = 1$. The left- and right-hand limits agree, but $f(1) = 0$, so there is a removable discontinuity at $x = 1$.

(c) We have $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0$ and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \sin(x - 2) = \sin(0) = 0$. Thus $\lim_{x \to 2} f(x) = 0$. Also, $f(2) = 2 - 2 = 0$. So f is continuous at $x = 2$.

8. Use the Intermediate Value Theorem to prove that there is a number c between 0 and $\frac{\pi}{2}$ satisfying the equation

$$c = \cos c.$$

Solution: Let $f(x) = \cos x - x$. Since both the cosine function and the function x are continuous everywhere, we know that f is continuous everywhere, as well. In particular, f is continuous on the interval $[0, \pi/2]$. We also find that

$$f(0) = \cos 0 - 0 = 1$$

and that

$$f(\pi/2) = \cos(\pi/2) - \pi/2 = -\pi/2.$$

Since 0 lies between 1 and $-\pi/2$, by the Intermediate Value theorem, we know there is a number c in the interval $(0, \pi/2)$ such that $f(c) = 0$. That is, $\cos c - c = 0$, or, equivalently, $\cos c = c$.