Math 101(04) Practice Hour Exam #2 – Solution 15 October 2001

1. Find $f'(x)$. Do not simplify.

   (a) $f(x) = (x^2 - 5)\sqrt{3x + 10}$.
   Solution: Using the product rule, we get

$$f'(x) = (x^2 - 5)\frac{1}{2}(3x + 10)^{-\frac{1}{2}} \cdot 3 + 2x\sqrt{3x + 10}.$$

   (b) $f(x) = \sqrt{x + \cos(x^2)}$.
   Solution: By the chain rule, we get

$$f'(x) = \frac{1}{2}(x + \cos(x^2))^{-\frac{1}{2}} \cdot (1 - \sin(x^2) \cdot 2x).$$

2. Let G, g, and h be functions, with $G = g \circ h$. Suppose h is given by $h(x) = x^2 - 3x$. Suppose further that $G'(1) = 4$. Find $G'(2)$.

Solution: By the chain rule, we know that

$$G'(2) \quad = \quad g'(h(2)) \cdot h'(2).$$

Since $h(x)$ is given as $x^2 - 3x$, we can easily find that $h(2) = 2^2 - 3 \cdot 2 = -2$ and that $h'(x) = 2x - 3$, so that $h'(2) = 1$. Thus we have

$$G'(2) \quad = \quad g'(-2) \cdot 1.$$

To determine $g'(-2)$, we try using the remaining given information. From $G'(1) = 4$, using the chain rule, we get

$$\begin{aligned} 4 &= G'(1) \\ &= g'(h(1)) \cdot h'(1) \\ &= g'(-2) \cdot (-1) \end{aligned}$$

So that $-g'(-2) = 4$, from which it follows that $g'(-2) = -4$, and so that $G'(2) = -4$.

3. Find an equation for the line tangent to the curve $y = (\cos(x))^2$ at the point where $x = \dfrac{\pi}{6}$. Leave your answer in exact form (in terms of π and square roots).

Solution: We have

$$\begin{aligned} y\left(\frac{\pi}{6}\right) &= \left(\cos\left(\frac{\pi}{6}\right)\right)^2 \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= \frac{3}{4}. \end{aligned}$$

To find the slope, we take

$$y' = -2\cos(x)\sin(x)$$

and evaluate at $x = \dfrac{\pi}{6}$. We get

$$\begin{aligned} y'\left(\frac{\pi}{6}\right) &= -2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

The equation for the line is

$$\left(y - \frac{3}{4}\right) = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right).$$

4. Find the slope of the line tangent to the curve $y^3 - 2xy^2 + x^4 = 1$ at the point $(1, 2)$.

Solution: To find $\dfrac{dy}{dx}$, we use implicit differentiation. We get

$$3y^2y' - 4xyy' - 2y^2 + 4x^3 = 0.$$

We solve for y', getting

$$y' = \frac{2y^2 - 4x^3}{3y^2 - 4xy}.$$

Evaluating this at the point $(1, 2)$, we get

$$\begin{aligned} y' &= \frac{4}{4} \\ &= 1. \end{aligned}$$

5. After 25 years of unrelieved boredom, the Viking lander on Mars picks up a small rock and hurls it directly upward. The rock's height h (in meters) at time t (in seconds) is given by

$$h(t) = 40t - 5t^2.$$

(a) What is the rock's velocity and direction of travel six seconds after it is thrown?

Solution: We have

$$h'(t) = 40 - 10t,$$

so that $h'(6) = -20$. Six seconds after it is thrown, the rock is moving downward at 20 meters per second.

(b) What is the maximum height reached by the rock?

Solution: The rock is at its maximum height when its velocity is 0. This occurs at $t = 4$. The height at $t = 4$ is

$$\begin{aligned} h(4) &= 40 \times 4 - 5 \times 16 \\ &= 80 \text{ meters.} \end{aligned}$$

6. The number-crunchers at the Shrink-to-Fit Sock Shop estimate that their company makes a profit P (in dollars) given by

$$P(x) = 2x - 0.002x^2 - 300$$

when they manufacture x pair of socks per month. Suppose that the current production level at Shrink-to-Fit is 600 pair of socks per month.

(a) What is the current monthly profit and the marginal profit? Be sure to include appropriate units.

Solution: The monthly profit is

$$\begin{aligned} P(600) &= 2 \times 600 - 0.002(600)^2 - 300 \\ &= 1200 - 720 - 300 \\ &= 180 \text{ dollars per month} \end{aligned}$$

The marginal profit is given by

$$P'(x) = 2 - 0.004x.$$

At $x = 600$, we get

$$\begin{aligned} P'(600) &= 2 - 0.004 \times 600 \\ &= -0.40. \end{aligned}$$

The marginal profit is -40 cents per pair of socks.

(b) What action should the company take to increase its profits? Be as specific as possible.

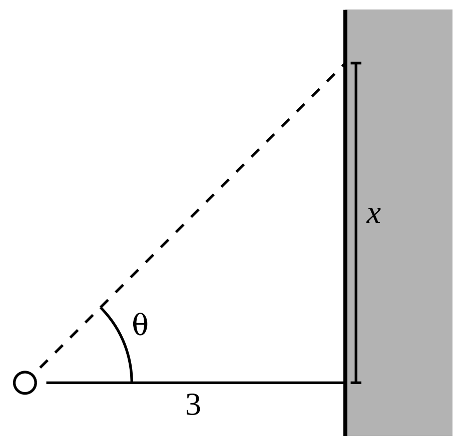
Solution: Since the marginal profit is negative, the company should decrease production to increase profit. It should continue to decrease production until the marginal profit reaches zero. The optimal production level x satisfies

$$2 - 0.004x = 0,$$

so the company should scale back production to 500 pair of socks per month.

7. A lighthouse stands on an island three miles away from a perfectly straight, north-south shoreline. The lighthouse beacon makes one complete rotation every five seconds. How fast is the image from the lighthouse beam moving along the shore when it is three miles north of the point on the shore closest to the lighthouse? Remember to include units.

Solution: As in the picture, let x be the distance between the image of the lighthouse beam and the point on the shore closest to the lighthouse. Let θ be the angle that the lighthouse beam makes with a line drawn directly to the shore. We are told that



$$\frac{d\theta}{dt} = \frac{2\pi}{5}$$

and asked to find $\frac{dx}{dt}$ when $x = 3$. From the picture, we find that $\tan\theta = \frac{x}{3}$. Differentiating both sides of this equation with respect to t, we get

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{3} \cdot \frac{dx}{dt}.$$

When $x = 3$, we find that $\theta = \frac{\pi}{4}$, so $\sec\theta = \sqrt{2}$. Thus when $x = 3$, our equation becomes

$$\begin{aligned} 2 \cdot \frac{d\theta}{dt} &= \frac{1}{3} \cdot \frac{dx}{dt}, \quad \text{so} \\ \frac{dx}{dt} &= 6\frac{d\theta}{dt}. \end{aligned}$$

Using the fact that $\frac{d\theta}{dt} = \frac{2\pi}{5}$, we find that

$$\frac{dx}{dt} = \frac{12\pi}{5} \frac{\text{miles}}{\text{second}}.$$