Math 101(04) **Practice Exam III– Solution** **5 November 2001**

1. Find the linear function $L(n)$ that best approximates $\tan((45+n)^\circ)$ when n is close to 0.

   Solution: Let $f(x) = \tan(x)$. Then

$$df = \sec^2(x)\,dx$$

   and at $x = \frac{\pi}{4}$ (that is, 45°), we get

$$\begin{aligned} df &= \sec^2\left(\frac{\pi}{4}\right)\,dx \\ &= 2\,dx. \end{aligned}$$

   The change from $x = 45^\circ$ to $x = (45+n)^\circ$ corresponds to a dx of n°. Since one degree is $\frac{\pi}{180}$ radians, we get

$$\begin{aligned} dx &= n\left(\frac{\pi}{180}\right) \\ &= \frac{n\pi}{180}. \end{aligned}$$

   Thus a change from $x = 45^\circ$ to $x = (45+n)^\circ$ causes a change in the value of $f(x)$ approximately equal to

$$\begin{aligned} df &= 2\,dx \\ &= 2\left(\frac{n\pi}{180}\right) \\ &= \frac{n\pi}{90}. \end{aligned}$$

   Since $f(45^\circ) = \tan(45^\circ) = 1$, our linear approximation is

$$\sin((45+n)^\circ) \approx 1 + \frac{n\pi}{90}.$$

2. A circular table top is measured to be 6 ft in diameter, using an old measuring tape whose markings could be off by as much as $\frac{1}{2}$ inch. Suppose this measurement is used to calculate the area of the table. Use differentials to estimate the largest possible error in the determination of the area.

Solution: The area of a circle is given by

$$\begin{aligned} A &= \pi r^2 \\ &= \frac{\pi D^2}{4} \end{aligned}$$

where D is the circle's diameter. Thus we get

$$dA = \frac{\pi}{2} D\, dD.$$

In this problem, we use $D = 6$ feet and $dD = 1/2$ inch $= 1/24$ foot. The largest possible error in the calculated area is thus

$$\begin{aligned} dA &= \frac{\pi}{2} \times 6 \times \frac{1}{24} \\ &= \frac{\pi}{8} \text{ ft}^2. \end{aligned}$$

(This turns out to be about 56.5 square inches.)

3. A water balloon is dropped from a hovering United Nations helicopter. The balloon's velocity v (measured in feet per second, in the downward direction) obeys the equation

$$v'(t) = 32 - 0.2v(t).$$

At $t = 4$ seconds, the balloon's velocity is 88 feet per second. Use differentials or a linear approximation to approximate the balloon's velocity at $t = 4.5$ seconds.

Solution: We know $v(4) = 88$, so we can find

$$\begin{aligned} v'(4) &= 32 - 0.2v(4) \\ &= 32 - 0.2 \times 88 \\ &= 14.4. \end{aligned}$$

In terms of differentials, this means that $dv = 14.4\,dt$ at $t = 4$, so the velocity changes at about 14.4 feet per second per second near $t = 4$. In the half second between $t = 4$ and $t = 4.5$, then, we have

$$\begin{aligned} dv &= 14.4 \times 0.5 \\ &= 7.2 \end{aligned}$$

so that $v(4.5)$ is approximately $88 + 7.2 = 95.2$ feet per second.

Alternatively, the we could construct a linear approximation $L(t)$. We get

$$\begin{aligned} L(t) &= v(4) + v'(4)(t-4) \\ &= 88 + 14.4(t-4). \end{aligned}$$

Using this approximation, we get

$$\begin{aligned} L(4.5) &= 88 + 14.4(4.5-4) \\ &= 88 + 14.4 \times 0.5 \\ &= 88 + 7.2 \\ &= 95.2 \text{ feet per second.} \end{aligned}$$

4. Let $f(x) = x^3 - 6x^2 + 9x - 4$. Find the absolute maximum and minimum values of f on the interval $[0, 5]$.

Solution: Since f is continuous everywhere, it is continuous on the given interval, so the absolute maximum and absolute minimum values must occur either at critical numbers or at the endpoints of the interval. To find the critical numbers, we compute

$$f'(x) = 3x^2 - 12x + 9.$$

Since f' is defined for every value of x, the only critical numbers will be those numbers x for which $f'(x) = 0$. We solve

$$\begin{aligned} 3(x^2 - 4x + 3) &= 0 \\ 3(x - 3)(x - 1) &= 0 \end{aligned}$$

and determine that $x = 1$ and $x = 3$ are critical numbers. To find the absolute maximum and minimum values of f on the interval $[0, 5]$, we evaluate f at the endpoints of the interval and the critical numbers inside the interval. We get

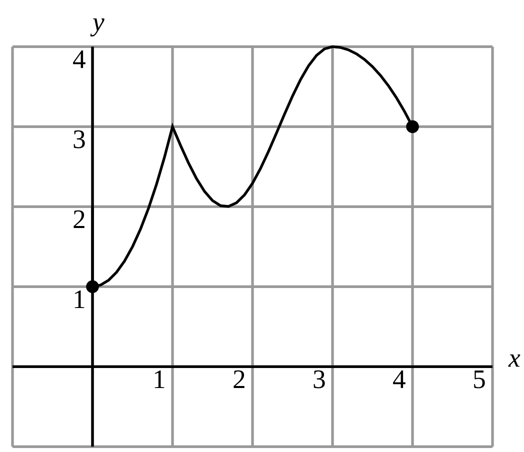
$$\begin{aligned} f(0) &= -4 \\ f(1) &= 0 \\ f(3) &= -4 \\ f(5) &= 16. \end{aligned}$$

The absolute minimum value is -4 and the absolute maximum value is 16.

5. Sketch the graph of a function f satisfying the following:

- f is continuous on $[0, 4]$
- f has a local maximum at 1
- f is not differentiable at 1, but is differentiable at all other points in $(0, 4)$
- f has an absolute maximum at 3
- the absolute minimum value of f is 1.

Here is one solution:



6. Let $f(x) = \cos x - 3x$.

(a) Prove that f has at least 1 root.

Solution: We find that

$$f(0) = \cos 0 - 0 = 1$$

and

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{3\pi}{2} = -\frac{3\pi}{2}.$$

Since both $\cos x$ and $3x$ are continuous on every interval, we know that f is continuous on every interval. In particular, f is continuous on $\left[0, \frac{\pi}{2}\right]$. Since 0 lies between $f(0)$ and $f\left(\frac{\pi}{2}\right)$, by the Intermediate Value Theorem, we know that there exists a number c in $\left(0, \frac{\pi}{2}\right)$ such that $f(c) = 0$. Thus f has at least one root.

(b) Prove that f has at most one root.

Solution: Suppose not. That is, suppose there are numbers a and b (with $a < b$, say) such that $f(a) = 0$ and $f(b) = 0$. Since $\cos x$ and $3x$ are both continuous and differentiable on the whole real line, we know that f is continuous and differentiable on any interval. In particular, f is continuous on $[a, b]$ and differentiable on (a, b). Thus by the Mean Value Theorem, there is a number c in (a, b) such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{0 - 0}{b - a} \\ &= 0. \end{aligned}$$

Now $f'(x) = -\sin x - 3$, so if $f'(c) = 0$, we must have

$$-\sin c - 3 = 0,$$

which implies that $\sin c = -3$. But this cannot happen, since the sine of any real number c must satisfy

$$-1 \leq \sin c \leq 1$$

and clearly -3 is not between -1 and 1. Thus we have a contradiction, and so our original assumption that f has two distinct roots must be false. Therefore f has at most one root.