Math 101(04) **Practice Exam IV – Solution** **26 November 2001**

1. Let f be the function given by $f(x) = x^4 + x^3 - 5x^2 + 6$.

   (a) List the intervals on which f is increasing and the intervals on which f is decreasing.

   Solution: We have

$$\begin{aligned} f'(x) &= 4x^3 + 3x^2 - 10x \\ &= x(4x^2 + 3x - 10) \\ &= x(4x - 5)(x + 2) \end{aligned}$$

   from which we get that the critical numbers for f are 0, -2, and $\frac{5}{4}$. chart for $f'(x)$:

	-2		0		$\frac{5}{4}$	
x:	$-$		$-$	$+$		$+$
$4x-5$:	$-$		$-$	$-$		$+$
$x+2$:	$-$		$+$	$+$		$+$
$f'(x)$:	$-$		$+$	$-$		$+$

   The function f is increasing on $[-2, 0]$ and $\left[\frac{5}{4}, \infty\right)$, and decreasing on $(-\infty, -2]$ and $\left[0, \frac{5}{4}\right]$.

   (b) List the values of x at which f has a local maximum and the values of x at which f has a local minimum.

   Solution: From the sign chart above, we see that f has a local minimum at $x = -2$ and another local minimum at $x = \frac{5}{4}$. The function has a local maximum at $x = 0$.

(c) List the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

Solution: We have $f''(x) = 12x^2 + 6x - 10$, so the second derivative of f can change sign only where $2(6x^2 + 3x - 5) = 0$, that is, where

$$x = \frac{-3 \pm \sqrt{129}}{12}.$$

We know that the graph of $f''(x)$ is a parabola opening upward (since the leading coefficient is positive), and because it has two real roots, we know that $f''(x)$ is negative between the two roots and positive elsewhere. Thus $f''(x)$ is positive for

$$x < \frac{-3 - \sqrt{129}}{12} \quad \text{or} \quad x > \frac{-3 + \sqrt{129}}{12}$$

and negative for

$$\frac{-3 - \sqrt{129}}{12} < x < \frac{-3 + \sqrt{129}}{12}.$$

The graph of f is concave upward on the intervals

$$\left(-\infty, \frac{-3 - \sqrt{129}}{12}\right) \quad \text{and} \quad \left(\frac{-3 + \sqrt{129}}{12}, \infty\right),$$

and concave downward on the interval

$$\left(\frac{-3 - \sqrt{129}}{12}, \frac{-3 + \sqrt{129}}{12}\right).$$

2. Let $f(x) = \dfrac{x\sqrt{3x^2+5}}{(2x+1)^2}$.

(a) Find $\lim_{x\to\infty} f(x)$.

Solution: We have

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x\sqrt{3x^2+5}}{4x^2+4x+1} \cdot \frac{(1/x)(1/x)}{(1/x^2)} \\
&= \lim_{x\to\infty} \frac{\sqrt{1/x^2}\sqrt{3x^2+5}}{4+\frac{4}{x}+\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{\sqrt{3+\frac{5}{x^2}}}{4+\frac{4}{x}+\frac{1}{x^2}} \\
&= \frac{\sqrt{3}}{4}.
\end{aligned}$$

(b) Find $\lim_{x\to-\infty} f(x)$.

Solution: We have

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x\sqrt{3x^2+5}}{4x^2+4x+1} \cdot \frac{(1/x)(1/x)}{(1/x^2)} \\
&= \lim_{x\to\infty} \frac{-\sqrt{1/x^2}\sqrt{3x^2+5}}{4+\frac{4}{x}+\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{-\sqrt{3+\frac{5}{x^2}}}{4+\frac{4}{x}+\frac{1}{x^2}} \\
&= -\frac{\sqrt{3}}{4}.
\end{aligned}$$

(c) Find $\lim_{x\to-\frac{1}{2}^+} f(x)$.

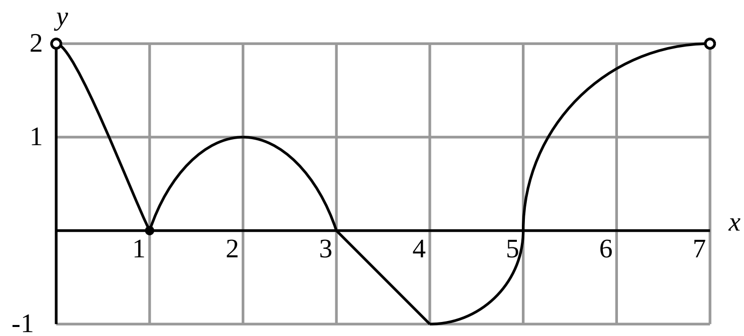
Solution: We have

$$\lim_{x\to-\frac{1}{2}^+} f(x) = \frac{-\frac{1}{2}\sqrt{\frac{3}{4}+5}}{0^+},$$

a negative number over a very tiny positive number, so that

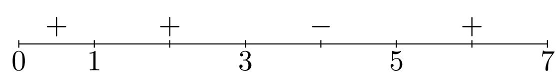
$$\lim_{x\to-\frac{1}{2}^+} f(x) = -\infty.$$

3. The function f is continuous and differentiable on the interval $[0, 6]$. Here is a graph of the *derivative* of f. That is, this graph shows the curve $y = f'(x)$.



(a) List the values of x at which f has a local maximum and the values of x at which f has a local minimum.

Solution: From the graph of f', we can see that the critical numbers of f are 1, 3, and 5. Here is a sign chart for f':



The function f has a local maximum at $x = 3$, and a local minimum at $x = 5$. The critical point at $x = 1$ is neither a maximum nor a minimum. (There is also an endpoint minimum at $x = 0$ and an endpoint maximum at $x = 7$.)

(b) On what intervals is the graph of f concave upward? On what intervals is the graph of f concave downward?

Solution: We know that the graph of f is concave upward on intervals where f' is increasing, and concave downward on intervals where f' is decreasing. From the picture, we see that f' is increasing, and therefore the graph of f is concave upward, on the intervals

$$(1, 2) \quad \text{and} \quad (4, 7).$$

Again from the picture, f' is decreasing, and therefore the graph of f is concave downward, on the intervals

$$(0, 1) \quad \text{and} \quad (2, 4).$$

4. The equation $x^3 + 4x + 4 = 0$ has a single root, near $x = -1$.

(a) Set up the iteration rule you would use in applying Newton's method to find the root of the equation above. Your rule should give a formula for x_{n+1} in terms of x_n.

Solution:

$$x_{n+1} = \boxed{x_n - \frac{x_n^3 + 4x_n + 4}{3x_n^2 + 4}}$$

(b) Using the initial guess $x_0 = -1$, find x_1. Leave your answer in exact form.

Solution: We have

$$\begin{aligned} x_1 &= -1 - \frac{(-1)^3 + 4(-1) + 4}{3(-1)^2 + 4} \\ &= -1 + \frac{1}{7} \\ &= -\frac{6}{7}. \end{aligned}$$

5. Find the positive number x for which $5x + \dfrac{1}{x^2}$ is as small as possible.

Solution: Let $f(x) = 5x + \dfrac{1}{x^2}$. We need to minimize $f(x)$ with $x > 0$. We get

$$f'(x) = 5 - \frac{2}{x^3}.$$

The only cricital number is the solution to

$$\begin{aligned} 5 &= \frac{2}{x^3} \\ x^3 &= \frac{2}{5}. \end{aligned}$$

It is easy to check that $f'(x)$ is negative for $x < \sqrt[3]{\frac{2}{5}}$ and $f'(x)$ is positive for $x > \sqrt[3]{\frac{2}{5}}$, so we have a local (and absolute) minimum. The function f is minimized when

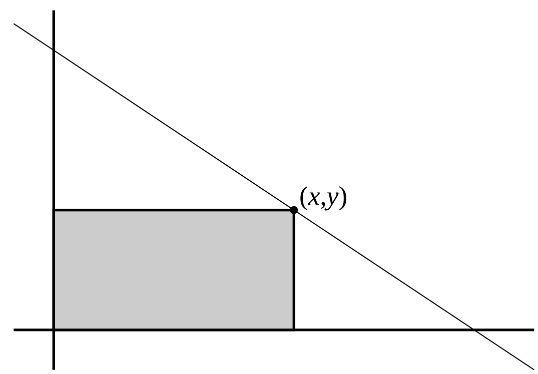
$$x = \sqrt[3]{\frac{2}{5}}.$$

6. One corner of a rectangle is at the origin in the xy-plane, and the opposite corner lies in the first quadrant, along the line $y = 7 - \frac{2}{3}x$. The sides of the rectangle are parallel to the coordinate axes. Find the largest possible area of such a rectangle.

Solution:

Let (x, y) be the coordinates of the rectangle's upper right corner. Then the area of the rectangle is given by

$$A = xy.$$



We also know that $y = 7 - \frac{2}{3}x$, so we can take

$$\begin{aligned} A &= x\left(7 - \frac{2}{3}x\right) \\ &= 7x - \frac{2}{3}x^2. \end{aligned}$$

We get

$$A' = 7 - \frac{4}{3}x$$

so that A has only one critical number at

$$x = \frac{21}{4}.$$

It is easy to see that A' is positive for values of x less than $\frac{21}{4}$ and negative for values of x greater than $\frac{21}{4}$, so that we have a local maximum at $x = \frac{21}{4}$. The area of the largest rectangle is

$$\frac{21}{4}\left(7 - \frac{2}{3}\frac{21}{4}\right) = \frac{147}{8}.$$