Limit problems from Exam 1 and Exam 4:

1. Find $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$.

   Solution: The given function is the same as $\dfrac{(x-2)(x+3)}{x-2}$, which is undefined at $x=2$, but elsewhere is equal to $x+3$. Its limit as $x \to 2$ must therefore be the same as $\lim_{x\to 2} x+3$, which is 5.

2. Find $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$.

   Solution: We multiply top and bottom by $\sqrt{x}+2$ to get

$$\frac{x-4}{(x-4)(\sqrt{x}+2)},$$

   which is undefined at $x=4$, but everywhere else is equal to $\dfrac{1}{\sqrt{x}+2}$. The limit as $x \to 4$ must therefore be the same as $\lim_{x\to 4} \dfrac{1}{\sqrt{x}+2}$, which is $\dfrac{1}{4}$.

3. Find $\lim_{x\to 3^-} \dfrac{x-5}{(x-3)(x-1)}$.

   Solution: The expression has a zero denominator at 3, so we suspect a limit of $\pm\infty$. We investigate

$$\frac{3^- - 5}{(3^- - 3)(3^- - 1)} \quad \approx \quad \frac{-2}{(0^-)(2)}.$$

   This is a negative number divided by a very tiny negative number, so the result is a large positive number. We conclude that

$$\lim_{x\to 3^-} \frac{x-5}{(x-3)(x-1)} \quad = \quad \infty.$$

4. $\lim_{x\to 5} \dfrac{x^2-x-20}{x-5}$.

   Solution: The given function is the same as $\dfrac{(x-5)(x+4)}{x-5}$, which is undefined at $x=5$, but elsewhere is equal to $x+4$. Its limit as $x \to 5$ must therefore be the same as $\lim_{x\to 5} x+4$, which is 9.

5. Find $\lim_{x \to 5^+} \frac{\sqrt{x+4}-4}{x-5}$.

Solution: The expression has a zero denominator (and a non-zero numerator) at 5, so we suspect a limit of $\pm\infty$. We investigate

$$\frac{\sqrt{5^+ + 4} - 4}{(5^+ - 5)} \quad \approx \quad \frac{3-4}{(0^+)}.$$

This is a negative number divided by a very tiny positive number, so the result is a large negative number. We conclude that

$$\lim_{x \to 5^+} \frac{\sqrt{x+4}-4}{x-5} \quad = \quad -\infty.$$

6. $\lim_{x \to 5} \frac{\sqrt{x+4}-3}{x-5}$.

Solution: We multiply top and bottom by $\sqrt{x+4}+3$ to get

$$\frac{x+4-9}{(x-5)(\sqrt{x+4}+3)} \quad = \quad \frac{x-5}{(x-5)(\sqrt{x+4}+3)},$$

which is undefined at $x = 5$, but everywhere else is equal to $\frac{1}{\sqrt{x+4}+3}$. The limit as $x \to 5$ must therefore be the same as $\lim_{x \to 5} \frac{1}{\sqrt{x+4}+3}$, which is $\frac{1}{6}$.

7. Let $f(x) = \frac{x\sqrt{3x^2+5}}{(2x+1)^2}$.

(a) Find $\lim_{x \to \infty} f(x)$.

Solution: We have

$$\begin{aligned}
\lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{x\sqrt{3x^2+5}}{4x^2+4x+1} \cdot \frac{(1/x)(1/x)}{(1/x^2)} \\
&= \lim_{x \to \infty} \frac{\sqrt{1/x^2}\sqrt{3x^2+5}}{4 + \frac{4}{x} + \frac{1}{x^2}} \\
&= \lim_{x \to \infty} \frac{\sqrt{3 + \frac{5}{x^2}}}{4 + \frac{4}{x} + \frac{1}{x^2}} \\
&= \frac{\sqrt{3}}{4}.
\end{aligned}$$

(b) Find $\lim_{x\to-\infty} f(x)$.
Solution: We have

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x\sqrt{3x^2+5}}{4x^2+4x+1}\cdot\frac{(1/x)(1/x)}{(1/x^2)} \\ &= \lim_{x\to\infty} \frac{-\sqrt{1/x^2}\sqrt{3x^2+5}}{4+\frac{4}{x}+\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{-\sqrt{3+\frac{5}{x^2}}}{4+\frac{4}{x}+\frac{1}{x^2}} \\ &= -\frac{\sqrt{3}}{4}.\end{aligned}$$

(c) Find $\lim_{x\to-\frac{1}{2}^+} f(x)$.
Solution: We have

$$\lim_{x\to-\frac{1}{2}^+} f(x) = \frac{-\frac{1}{2}\sqrt{\frac{3}{4}+5}}{0^+},$$

a negative number over a very tiny positive number, so that

$$\lim_{x\to-\frac{1}{2}^+} f(x) = -\infty.$$

8. Let $f(x) = \dfrac{2x^3+4}{|x|(3x-1)^2}$.

(a) Find $\lim_{x\to\infty} f(x)$.
Solution: We have

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x^3+4}{x(9x^2-6x+1)} \\ &= \lim_{x\to\infty} \frac{2x^3+4}{9x^3-6x^2+x} \\ &= \lim_{x\to\infty} \frac{2+\frac{4}{x^3}}{9-\frac{6}{x}+\frac{1}{x^2}} \\ &= \frac{2}{9}.\end{aligned}$$

(b) Find $\lim\limits_{x\to-\infty} f(x)$.

Solution: We have

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{2x^3+4}{(-x)(9x^2-6x+1)}\right) &= -\lim_{x\to\infty}\frac{2x^3+4}{9x^3-6x^2+x} \\
&= -\lim_{x\to\infty}\frac{2+\frac{4}{x^3}}{9-\frac{6}{x}+\frac{1}{x^2}} \\
&= -\frac{2}{9}.
\end{aligned}$$

(c) Find $\lim\limits_{x\to 0^-} f(x)$.

Solution: We have

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \frac{2(0)^3+4}{0^+(3(0)-1)^2} \\
&= \frac{4}{1\cdot 0^+},
\end{aligned}$$

a positive number over a very tiny positive number, so that

$$\lim_{x\to 0^-} f(x) = \infty.$$

9. Find $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+x}-x}$.

Solution: The form is $\dfrac{1}{\infty-\infty}$, so we need to do some manipulation. We multiply top and bottom by $\sqrt{x^2+x}+x$ to get

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{x^2+x}+x}{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)} &= \lim_{x\to\infty}\frac{\sqrt{x^2+x}+x}{x^2+x-x^2} \\
&= \lim_{x\to\infty}\frac{\sqrt{x^2+x}+x}{x} \\
&= \lim_{x\to\infty}\left(\sqrt{\frac{x^2+x}{x^2}}+1\right) \\
&= \lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x}}+1\right) \\
&= 2.
\end{aligned}$$