1. The Whizzo Confectionery Company is planning to market a new candy called the Dental Delight. It consists of a (spherical) steel ball with radius 5 mm, covered with a milk chocolate coating $\frac{1}{4}$ mm thick.

   Use differentials to approximate the volume of milk chocolate in each Dental Delight. Be sure to include units in your answer.

   (The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

   Solution: We have

$$dV = 4\pi r^2,$$

   so when $r = 5$ mm, we get

$$\begin{aligned} dV &= 4\pi(25)\ \text{mm}^2\, dr \\ &= 100\pi \times \frac{1}{4}\ \text{mm}^3 \\ &= 25\pi\ \text{mm}^3. \end{aligned}$$

2. A circular table top is measured to be 6 ft in diameter, using an old measuring tape whose markings could be off by as much as $\frac{1}{2}$ inch. Suppose this measurement is used to calculate the area of the table. Use differentials to estimate the largest possible error in the determination of the area.

   Solution: The area of a circle is given by

$$\begin{aligned} A &= \pi r^2 \\ &= \frac{\pi D^2}{4} \end{aligned}$$

   where D is the circle's diameter. Thus we get

$$dA = \frac{\pi}{2} D\, dD.$$

   In this problem, we use $D = 6$ feet and $dD = 1/2$ inch $= 1/24$ foot. The largest possible error in the calculated area is thus

$$\begin{aligned} dA &= \frac{\pi}{2} \times 6 \times \frac{1}{24} \\ &= \frac{\pi}{8}\ \text{ft}^2. \end{aligned}$$

   (This turns out to be about 56.5 square inches.)

3. Find the local linearization $L(x)$ for the function $f(x) = \dfrac{1}{1+x^2}$ at $a = 3$.

Solution: We have

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

so that

$$\begin{aligned} f'(3) &= -\frac{6}{10^2} \\ &= -\frac{6}{100}. \end{aligned}$$

We also compute

$$f(3) = \frac{1}{10},$$

and conclude that

$$L(x) = \frac{1}{10} - \frac{6}{100}(x-3).$$

4. Find the linear function $L(n)$ that best approximates $\tan((45+n)^\circ)$ when n is close to 0.

Solution: Let $f(x) = \tan(x)$. Then

$$df = \sec^2(x)\, dx$$

and at $x = \dfrac{\pi}{4}$ (that is, 45°), we get

$$\begin{aligned} df &= \sec^2\left(\frac{\pi}{4}\right) dx \\ &= 2\, dx. \end{aligned}$$

The change from $x = 45^\circ$ to $x = (45+n)^\circ$ corresponds to a dx of n°. Since one degree is $\dfrac{\pi}{180}$ radians, we get

$$\begin{aligned} dx &= n\left(\frac{\pi}{180}\right) \\ &= \frac{n\pi}{180}. \end{aligned}$$

Thus a change from $x = 45^\circ$ to $x = (45+n)^\circ$ causes a change in the value of $f(x)$ approximately equal to

$$\begin{aligned} df &= 2\,dx \\ &= 2\left(\frac{n\pi}{180}\right) \\ &= \frac{n\pi}{90}. \end{aligned}$$

Since $f(45^\circ) = \tan(45^\circ) = 1$, our linear approximation is

$$\sin((45+n)^\circ) \approx 1 + \frac{n\pi}{90}.$$

5. In a computer model of bacterial growth, the function $P(t)$ represents the population of a certain bacterium at time t. It is known that $P(10) = 5000$ and that, in general, the rate of growth in the population follows the rule

$$P'(t) = 0.02P(t).$$

Use a linear approximation to estimate $P(10.5)$.

Solution: We know $P(10) = 5000$, and we can find $P'(10)$, because we know that

$$\begin{aligned} P'(10) &= 0.02P(10) \\ &= 0.02(5000) \\ &= 100. \end{aligned}$$

The linear approximation $L(t)$ is thus

$$L(t) = 5000 + 100(t-10),$$

and so we get

$$\begin{aligned} P(10.5) &\approx 5000 + 100(10.5-10) \\ &= 5050. \end{aligned}$$

At $t = 10.5$, we expect the population will be approximately 5050.

6. A water balloon is dropped from a hovering United Nations helicopter. The balloon's velocity v (measured in feet per second, in the downward direction) obeys the equation

$$v'(t) = 32 - 0.2v(t).$$

At $t = 4$ seconds, the balloon's velocity is 88 feet per second. Use differentials or a linear approximation to approximate the balloon's velocity at $t = 4.5$ seconds.

Solution: We know $v(4) = 88$, so we can find

$$\begin{aligned} v'(4) &= 32 - 0.2v(4) \\ &= 32 - 0.2 \times 88 \\ &= 14.4. \end{aligned}$$

In terms of differentials, this means that $dv = 14.4\,dt$ at $t = 4$, so the velocity changes at about 14.4 feet per second per second near $t = 4$. In the half second between $t = 4$ and $t = 4.5$, then, we have

$$\begin{aligned} dv &= 14.4 \times 0.5 \\ &= 7.2 \end{aligned}$$

so that $v(4.5)$ is approximately $88 + 7.2 = 95.2$ feet per second.

Alternatively, the we could construct a linear approximation $L(t)$. We get

$$\begin{aligned} L(t) &= v(4) + v'(4)(t - 4) \\ &= 88 + 14.4(t - 4). \end{aligned}$$

Using this approximation, we get

$$\begin{aligned} L(4.5) &= 88 + 14.4(4.5 - 4) \\ &= 88 + 14.4 \times 0.5 \\ &= 88 + 7.2 \\ &= 95.2 \text{ feet per second.} \end{aligned}$$