1. The Whizzo Confectionery Company is planning to market a new candy called the Dental Delight. It consists of a (spherical) steel ball with radius 5 mm, covered with a milk chocolate coating \( \frac{1}{4} \) mm thick.

Use differentials to approximate the volume of milk chocolate in each Dental Delight. Be sure to include units in your answer.

(The volume \( V \) of a sphere of radius \( r \) is given by \( V = \frac{4}{3} \pi r^3 \).)

Solution: We have

\[
dV = 4\pi r^2,
\]

so when \( r = 5 \) mm, we get

\[
dV = 4\pi(25) \text{ mm}^2 \, dr
\]

\[
= 100\pi \times \frac{1}{4} \text{ mm}^3
\]

\[
= 25\pi \text{ mm}^3.
\]

2. A circular table top is measured to be 6 ft in diameter, using an old measuring tape whose markings could be off by as much as \( \frac{1}{2} \) inch. Suppose this measurement is used to calculate the area of the table. Use differentials to estimate the largest possible error in the determination of the area.

Solution: The area of a circle is given by

\[
A = \pi r^2
\]

\[
= \pi D^2
\]

where \( D \) is the circle’s diameter. Thus we get

\[
dA = \frac{\pi}{2} D dD.
\]

In this problem, we use \( D = 6 \) feet and \( dD = 1/2 \) inch = 1/24 foot. The largest possible error in the calculated area is thus

\[
dA = \frac{\pi}{2} \times 6 \times \frac{1}{24}
\]

\[
= \frac{\pi}{8} \text{ ft}^2.
\]

(This turns out to be about 56.5 square inches.)
3. Find the local linearization $L(x)$ for the function $f(x) = \frac{1}{1 + x^2}$ at $a = 3$.

Solution: We have

$$f'(x) = -\frac{2x}{(1 + x^2)^2}$$

so that

$$f'(3) = -\frac{6}{10^2} = -\frac{6}{100}.$$ 

We also compute

$$f(3) = \frac{1}{10},$$

and conclude that

$$L(x) = \frac{1}{10} - \frac{6}{100}(x - 3).$$

4. Find the linear function $L(n)$ that best approximates $\tan((45 + n)^\circ)$ when $n$ is close to 0.

Solution: Let $f(x) = \tan(x)$. Then

$$df = \sec^2(x) \, dx$$

and at $x = \frac{\pi}{4}$ (that is, $45^\circ$), we get

$$df = \sec^2 \left( \frac{\pi}{4} \right) \, dx = 2 \, dx.$$ 

The change from $x = 45^\circ$ to $x = (45 + n)^\circ$ corresponds to a $dx$ of $n^\circ$. Since one degree is $\frac{\pi}{180}$ radians, we get

$$dx = n \left( \frac{\pi}{180} \right) = \frac{n\pi}{180}.$$
Thus a change from \( x = 45^\circ \) to \( x = (45 + n)^\circ \) causes a change in the value of \( f(x) \) approximately equal to

\[
\frac{df}{dx} = 2 \left( \frac{n \pi}{180} \right) = \frac{n \pi}{90}.
\]

Since \( f(45^\circ) = \tan(45^\circ) = 1 \), our linear approximation is

\[
\sin((45 + n)^\circ) \approx 1 + \frac{n \pi}{90}.
\]

5. In a computer model of bacterial growth, the function \( P(t) \) represents the population of a certain bacterium at time \( t \). It is known that \( P(10) = 5000 \) and that, in general, the rate of growth in the population follows the rule

\[
P'(t) = 0.02P(t).
\]

Use a linear approximation to estimate \( P(10.5) \).

Solution: We know \( P(10) = 5000 \), and we can find \( P'(10) \), because we know that

\[
P'(10) = 0.02P(10) = 0.02(5000) = 100.
\]

The linear approximation \( L(t) \) is thus

\[
L(t) = 5000 + 100(t - 10),
\]

and so we get

\[
P(10.5) \approx 5000 + 100(10.5 - 10) = 5050.
\]

At \( t = 10.5 \), we expect the population will be approximately 5050.

6. A water balloon is dropped from a hovering United Nations helicopter. The balloon’s velocity \( v \) (measured in feet per second, in the downward direction) obeys the equation

\[
v'(t) = 32 - 0.2v(t).
\]
At $t = 4$ seconds, the balloon’s velocity is 88 feet per second. Use differentials or a linear approximation to approximate the balloon’s velocity at $t = 4.5$ seconds.

Solution: We know $v(4) = 88$, so we can find

$$v'(4) = 32 - 0.2v(4)$$
$$= 32 - 0.2 \times 88$$
$$= 14.4.$$

In terms of differentials, this means that $dv = 14.4\, dt$ at $t = 4$, so the velocity changes at about 14.4 feet per second per second near $t = 4$. In the half second between $t = 4$ and $t = 4.5$, then, we have

$$dv = 14.4 \times 0.5$$
$$= 7.2$$

so that $v(4.5)$ is approximately $88 + 7.2 = 95.2$ feet per second.

Alternatively, the we could construct a linear approximation $L(t)$. We get

$$L(t) = v(4) + v'(4)(t - 4)$$
$$= 88 + 14.4(t - 4).$$

Using this approximation, we get

$$L(4.5) = 88 + 14.4(4.5 - 4)$$
$$= 88 + 14.4 \times 0.5$$
$$= 88 + 7.2$$
$$= 95.2 \text{ feet per second.}$$