1. Show that the equation \( x^5 + 10x + 3 = 0 \) has exactly one real root.

Solution: Let \( f(x) = x^5 + 10x + 3 \). Then \( f(-1) = -8 \) and \( f(0) = 3 \). Since 0 is between \(-8\) and 3, and \( f \) is continuous everywhere (because it’s a polynomial), the Intermediate Value Theorem guarantees that there is some number \( c \) in \((-1, 0)\) for which \( f(c) = 0 \). That is, the given equation has at least one root.

Now suppose that the equation has two distinct roots. Let \( a \) and \( b \) denote the two roots, so that \( f(a) = f(b) = 0 \). The function \( f \) is continuous and differentiable everywhere, so we can use the Mean Value Theorem to conclude that there is some number \( c \) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a} = 0.
\]

On the other hand, \( f'(x) = 5x^4 + 10 \), and since \( x^4 \geq 0 \) for all \( x \), we conclude that \( f'(x) \geq 10 \) for all \( x \). This contradicts the existence of a number \( c \) for which \( f'(c) = 0 \), so we must conclude that there are not two distinct roots. Thus there is at most one solution to the given equation.

2. Show that the equation \( \cos x = 2x \) has at most one solution.

Solution: Let \( f(x) = \cos x - 2x \). We need to show that \( f \) has at most one root. Suppose not. Then there are two distinct numbers, \( a \) and \( b \), such that \( f(a) = f(b) = 0 \). The function \( f \) is continuous and differentiable everywhere, so we can invoke the Mean Value Theorem to conclude that there is a number \( c \) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a} = 0.
\]

However, \( f'(x) = -\sin x - 2 \), so if \( f'(c) = 0 \), then we get

\[
 0 = f'(c) = -\sin c - 2,
\]

which implies that \( \sin c = -2 \). Since \( \sin x \) lies in the interval \([-1, 1]\) for every real number \( x \), we have a contradiction. There can be at most one solution to the given equation.

3. Show that the equation \( x^4 + 4x + c = 0 \) has at most two real roots.

Solution: Let \( f(x) = x^4 + 4x + c \). Note that since \( f \) is a polynomial, it is continuous and differentiable everywhere, as are all of its derivatives.

Suppose that \( f \) has three distinct roots, \( a_1, a_2, \) and \( a_3 \). Choose the names so that \( a_1 < a_2 < a_3 \). Since \( f \) is continuous on \([a_1, a_2]\) and differentiable on \((a_1, a_2)\), by the Mean Value Theorem, there is a number \( b_1 \) in \((a_1, a_2)\), such that

\[
 f'(b_1) = \frac{f(a_2) - f(a_1)}{a_2 - a_1} = 0.
\]
Similarly, since \( f \) is continuous on \([a_2, a_3]\) and differentiable on \((a_2, a_3)\), by the Mean Value Theorem, there is a number \( b_2 \) in \((a_2, a_3)\), such that

\[
f'(b_2) = \frac{f(a_3) - f(a_2)}{a_3 - a_2} = 0.
\]

Taking a derivative of \( f \), we get

\[
f'(x) = 4x^3 + 4
\]

We can solve the equation \( 4x^3 + 4 = 0 \); we get

\[
4x^3 = -1
\]
\[
x^3 = -1
\]
\[
x = -1.
\]

Now since \( b_1 \) and \( b_2 \) are solutions to \( f'(x) = 0 \), and there is only one solution to \( f'(x) = 0 \), we must have \( b_1 = -1 \) and \( b_2 = -1 \). Thus \( b_1 = b_2 \). But this is a contradiction, because \( b_1 \) is in \((a_1, a_2)\) and \( b_2 \) is in \((a_2, a_3)\), so \( b_1 < a_2 < b_2 \), which implies \( b_1 < b_2 \).

Since our assumption that \( f \) has three roots led to a contradiction, we conclude that \( f \) has at most two roots.