1. Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.

   Solution: Let $f(x) = x^5 + 10x + 3$. Then $f(-1) = -8$ and $f(0) = 3$. Since 0 is between -8 and 3, and f is continuous everywhere (becuase it's a polynomial), the Intermediate Value Theorem guarantees that there is some number c in $(-1, 0)$ for which $f(c) = 0$. That is, the given equation has at least one root.
   Now suppose that the equation has two distinct roots. Let a and b denote the two roots, so that $f(a) = f(b) = 0$. The function f is continuous and differentiable everywhere, so we can use the Mean Value Theorem to conclude that there is some number c in (a, b) such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

   On the other hand, $f'(x) = 5x^4 + 10$, and since $x^4 \geq 0$ for all x, we conclude that $f'(x) \geq 10$ for all x. This contradicts the existence of a number c for which $f'(c) = 0$, so we must conclude that there are not two distinct roots. Thus there is at most one solution to the given equation.

2. Show that the equation $\cos x = 2x$ has at most one solution.

   Solution: Let $f(x) = \cos x - 2x$. We need to show that f has at most one root. Suppose not. Then there are two distinct numbers, a and b, such that $f(a) = f(b) = 0$. The function f is continuous and differentiable everywhere, so we can invoke the Mean Value Theorem to conclude that there is a number c in (a, b) for which

   $$f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

   However, $f'(x) = -\sin x - 2$, so if $f'(c) = 0$, then we get

   $$\begin{aligned} 0 &= f'(c) \\ &= -\sin c - 2, \end{aligned}$$

   which implies that $\sin c = -2$. Since $\sin x$ lies in the interval $[-1, 1]$ for every real number x, we have a contradiction. There can be at most one solution to the given equation.

3. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

   Solution: Let $f(x) = x^4 + 4x + c$. Note that since f is a polynomial, it is continuous and differentiable everywhere, as are all of its derivatives.
   Suppose that f has three distinct roots, a_1, a_2, and a_3. Choose the names so that $a_1 < a_2 < a_3$. Since f is continous on $[a_1, a_2]$ and differentiable on (a_1, a_2), by the Mean Value Theorem, there is a number b_1 in (a_1, a_2), such that

   $$f'(b_1) = \frac{f(a_2) - f(a_1)}{a_2 - a_1} = 0.$$

Similarly, since f is continous on $[a_2, a_3]$ and differentiable on (a_2, a_3), by the Mean Value Theorem, there is a number b_2 in (a_2, a_3), such that

$$f'(b_2) = \frac{f(a_3) - f(a_2)}{a_3 - a_2} = 0.$$

Taking a derivative of f, we get

$$f'(x) \quad = \quad 4x^3 + 4$$

We can solve the equation $4x^3 + 4 = 0$; we get

$$\begin{aligned} 4x^3 &= -1 \\ x^3 &= -1 \\ x &= -1. \end{aligned}$$

Now since b_1 and b_2 are solutions to $f'(x) = 0$, and there is only one solution to $f'(x) = 0$, we must have $b_1 = -1$ and $b_2 = -1$. Thus $b_1 = b_2$. But this is a contradiction, because b_1 is in (a_1, a_2) and b_2 is in (a_2, a_3), so $b_1 < a_2 < b_2$, which implies $b_1 < b_2$.
Since our assumption that f has three roots led to a contradiction, we conclude that f has at most two roots.