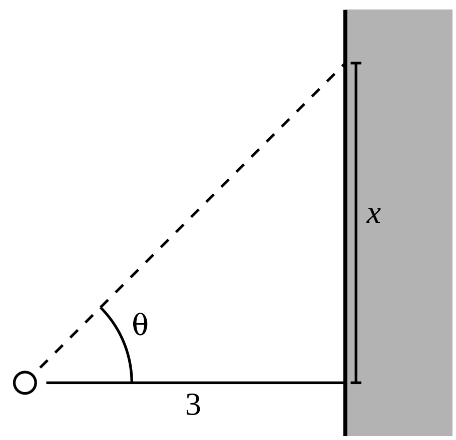
1. A lighthouse stands on an island three miles away from a perfectly straight, north-south shoreline. The lighthouse beacon makes one complete rotation every five seconds. How fast is the image from the lighthouse beam moving along the shore when it is three miles north of the point on the shore closest to the lighthouse? Remember to include units.

Solution: As in the picture, let x be the distance between the image of the lighthouse beam and the point on the shore closest to the lighthouse. Let θ be the angle that the lighthouse beam makes with a line drawn directly to the shore. We are told that



$$\frac{d\theta}{dt} = \frac{2\pi}{5}$$

and asked to find $\frac{dx}{dt}$ when $x = 3$. From the picture, we find that $\tan\theta = \frac{x}{3}$. Differentiating both sides of this equation with respect to t, we get

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{3} \cdot \frac{dx}{dt}.$$

When $x = 3$, we find that $\theta = \frac{\pi}{4}$, so $\sec\theta = \sqrt{2}$. Thus when $x = 3$, our equation becomes

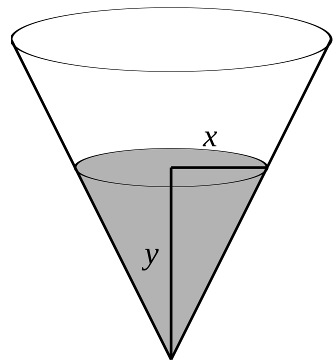
$$\begin{aligned} 2 \cdot \frac{d\theta}{dt} &= \frac{1}{3} \cdot \frac{dx}{dt}, \quad \text{so} \\ \frac{dx}{dt} &= 6\frac{d\theta}{dt}. \end{aligned}$$

Using the fact that $\frac{d\theta}{dt} = \frac{2\pi}{5}$, we find that

$$\frac{dx}{dt} = \frac{12\pi}{5} \frac{\text{miles}}{\text{second}}.$$

2. Concrete is being poured into a conical tank (vertex down) at the rate of 2 cubic feet per second. The tank is 10 feet deep and has a radius at the top of 5 feet. At what rate is the level of concrete in the tank rising when it has been filled to a depth of 6 feet? (The volume of a cone with radius r and height h is given by $\frac{1}{3}\pi r^2 h$).

Solution: As in the drawing, let x and y denote the depth and radius of the already-poured concrete. By similar triangles, we find that $x = \frac{y}{2}$, so we can express the volume V of the already-poured concrete as



$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y \\ &= \frac{\pi}{12}y^3. \end{aligned}$$

Differentiating both sides with respect to t, we find that

$$\begin{aligned} \frac{dV}{dt} &= \frac{3\pi}{12}y^2\frac{dy}{dt} \\ &= \frac{\pi y^2}{4}\frac{dy}{dt}. \end{aligned}$$

When $y = 6$, the equation becomes

$$\frac{dV}{dt} = 9\pi\frac{dy}{dt}.$$

We are given that $\frac{dV}{dt} = 2$, so we can solve for $\frac{dy}{dt}$ (which is what we want). We get

$$\frac{dy}{dt} = \frac{2}{9\pi} \text{ feet per second.}$$