Math 203(01) **Hour Exam I Solution** **12 October 2001**

1. Let $\vec{v} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and $\vec{w} = \vec{\imath} - 4\vec{\jmath} + \vec{k}$.

   (a) Find the cosine of the angle θ between $\vec{v}$ and $\vec{w}$.

   Solution: We have

$$\begin{aligned} ||\vec{v}|| &= \sqrt{2^2 + (-1)^2 + 3^2} \\ &= \sqrt{14}, \\ ||\vec{w}|| &= \sqrt{1^2 + (-4)^2 + 1^2} \\ &= \sqrt{18}, \text{ and} \\ \vec{v} \cdot \vec{w} &= 2 + 4 + 3 \\ &= 9. \end{aligned}$$

   Using these values and the fact that $\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}|| \cos\theta$, we get

$$\begin{aligned} \cos\theta &= \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||\,||\vec{w}||} \\ &= \frac{9}{\sqrt{14 \cdot 18}} \\ &= \frac{9}{\sqrt{252}} \\ &= \frac{3}{2\sqrt{7}}. \end{aligned}$$

   (b) Let $\vec{r} = \lambda\vec{\imath} + \vec{\jmath} - \lambda\vec{k}$. Find a value of λ that makes $\vec{r}$ perpendicular to $\vec{v}$.

   Solution: We need to find λ to make $\vec{v} \cdot \vec{r} = 0$. That is, we need to solve

$$\begin{aligned} 0 &= 2\lambda - 1 - 3\lambda \\ &= -\lambda - 1. \end{aligned}$$

   The solution is $\lambda = -1$.

2. Let A be the point $(1,1,3)$, B be the point $(4,0,5)$, and C be the point $(-4,3,-1)$.

   (a) Find an equation for the plane containing the points A, B, and C.
   Solution: We begin with two vectors

$$\begin{aligned}\overrightarrow{AB} &= 3\vec{\imath} - \vec{\jmath} + 2\vec{k} \text{ and} \\ \overrightarrow{AC} &= -5\vec{\imath} + 2\vec{\jmath} - 4\vec{k}.\end{aligned}$$

   The cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ will give us a normal vector to the plane. We get

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -1 & 2 \\ -5 & 2 & -4 \end{vmatrix} \\ &= 2\vec{\jmath} + \vec{k}.\end{aligned}$$

   An equation for the plane is

$$2(y-1) + (z-3) = 0.$$

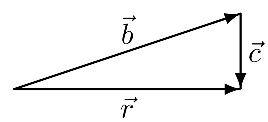
   (b) Find the area of triangle ABC.
   Solution: The area of the triangle is half the magnitude of the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$. We get

$$\begin{aligned}\text{Area} &= \frac{\sqrt{2^2 + 1^2}}{2} \\ &= \frac{\sqrt{5}}{2}.\end{aligned}$$

3. The mighty Australivec River is two miles wide, and flows due south with a constant current velocity of 3 miles per hour. A diplomat needs to cross from the west bank to a point directly across on the east bank. Her boat has a small, environmentally-friendly electric motor that propels the craft at a speed (in still water) of 6 miles per hour.

   (a) What course should diplomat steer to arrive on the east bank directly across from her starting point on the west bank?

   Solution: We take $\vec{\jmath}$ as the direction north and $\vec{\imath}$ as the direction east.



   The current is given by $\vec{c} = -3\vec{\jmath}$. Let

$$\vec{b} \;=\; b_1\vec{\imath} + b_2\vec{\jmath}$$

   represent the velocity of the boat in still water. Let $\vec{r}$ be the velocity vector of the boat crossing the river. We want the $\vec{\jmath}$ component of $\vec{r}$ to be zero, and we know that

$$\begin{aligned} \vec{r} &= \vec{c} + \vec{b} \\ &= b_1\vec{\imath} + (b_2 - 3)\vec{\jmath}. \end{aligned}$$

   From this it follows that $b_2 = 3$, and since $b_1^2 + b_2^2 = 36$, we find that $b_1 = \sqrt{27} = 3\sqrt{3}$. The angle θ between $\vec{r}$ and $\vec{b}$ is given by

$$\begin{aligned} \sin\theta &= \frac{||\vec{c}||}{||\vec{b}||} \\ &= \frac{3}{6}, \end{aligned}$$

   so that $\theta = \dfrac{\pi}{6}$, or 30 degrees. The proper course to steer is 30° north of east.

   (b) How long will it take her to cross the river?

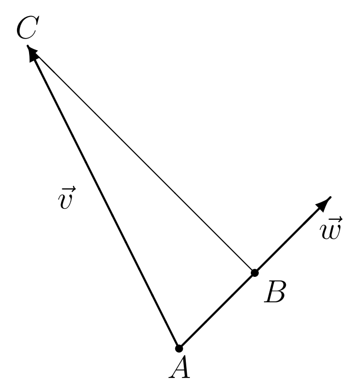
   Solution: The "ground speed" of the boat is the length of $\vec{r}$, which is exactly b_1, or $3\sqrt{3}$ miles per hour. The river is two miles wide, so the transit time is

$$\frac{2}{3\sqrt{3}} \;\approx\; 0.3849 \text{ hours}$$

   which is just over 23 minutes.

4. Let $\vec{v} = 6\vec{\imath} - 6\vec{\jmath} + 7\vec{k}$ and $\vec{w} = 3\vec{\jmath} + 4\vec{k}$. Suppose we have representations of $\vec{v}$ and $\vec{w}$ arranged as in the diagram below. The angle at point B is a right angle. Otherwise, the diagram is not necessarily drawn to scale.

   Write the vector $\overrightarrow{BC}$ in components (that is, in the form $x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ for some numbers x, y, and z).



Solution: If we think of line segment $\overline{AB}$ as a vector, it is simply the component of $\vec{v}$ parallel to $\vec{w}$. Its length is given by

$$\begin{aligned} \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||} &= \frac{-18 + 28}{5} \\ &= 2, \end{aligned}$$

so the vector $\overrightarrow{AB}$ in equal to

$$\frac{2\vec{w}}{||\vec{w}||} = \frac{6}{5}\vec{\jmath} + \frac{8}{5}\vec{k}.$$

Since $\overrightarrow{AB} + \overrightarrow{BC} = \vec{v}$, we have

$$\begin{aligned} \overrightarrow{BC} &= \vec{v} - \overrightarrow{AB} \\ &= (6\vec{\imath} - 6\vec{\jmath} + 7\vec{k}) - \left(\frac{6}{5}\vec{\jmath} + \frac{8}{5}\vec{k}\right) \\ &= 6\vec{\imath} - \frac{36}{5}\vec{\jmath} + \frac{27}{5}\vec{k}. \end{aligned}$$

5. Let $f(x,y) = e^{-x}\sqrt{16+y^2}$.

(a) Find the differential of f.

Solution: We have

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= (-e^{-x}\sqrt{16+y^2})\,dx + \frac{ye^{-x}}{\sqrt{16+y^2}}\,dy. \end{aligned}$$

(b) Note that $f(0,3) = 5$. Use the differential to estimate the value of $f(0.1, 2.95)$.

Solution: First we need the differential of f evaluated at the point $(0,3)$. We get

$$df(0,3) = -5\,dx + \frac{3}{5}\,dy.$$

We have $dx = 0.1$ and $dy = -0.05$. Thus

$$\begin{aligned} df &= -5(0.1) + \frac{3}{5}(-0.05) \\ &= -0.53. \end{aligned}$$

We estimate that $f(0.1, 2.95) \approx 5 - 0.53 = 4.47$.

6. Let $f(x, y, z) = x^2 - \frac{y^2}{4} - z^2$.

(a) Sketch the level surface $f(x, y, z) = 5$. Identify all the points where the surface intersects the coordinate axes. Also give the full name of the surface.

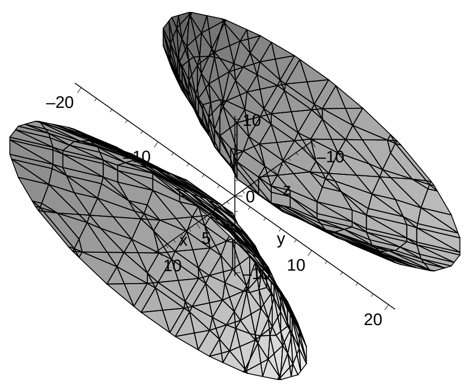
Solution: We are looking for the surface whose equation is

$$x^2 - \frac{y^2}{4} - z^2 = 5.$$

We rewrite this equation as

$$\frac{y^2}{4} + z^2 = x^2 - 5.$$

This says that the sections parallel to the yz-plane are ellipses, with longer axes in the y direction than in the z direction. The level curves are empty for $|x| < \sqrt{5}$, and are single points when $x = \pm\sqrt{5}$. Thus the only coordinate intercepts are $(\pm\sqrt{5}, 0, 0)$. The surface is an elliptic hyperboloid of two sheets. Here's a sketch:



(b) Find an equation for the plane tangent to the surface $f(x, y, z) = 5$ at the point $(5, 4, 4)$.

Solution: We have $\nabla f = 2x\vec{\imath} - \frac{y}{2}\vec{\jmath} - 2z\vec{k}$, and we get a normal vector to the tangent plane by evaluating

$$\nabla f(5, 4, 4) = 10\vec{\imath} - 2\vec{\jmath} - 8\vec{k}.$$

Using this vector and the point $(5, 4, 4)$, we can write an equation for the plane as

$$10(x - 5) - 2(y - 4) - 8(z - 4) = 0.$$

7. Let $h(x, y, z) = \dfrac{x^3 y}{1 + z^2}$.

(a) Let $\vec{u} = \dfrac{1}{\sqrt{17}}(2\vec{\imath} - 3\vec{\jmath} - 2\vec{k})$. Find $h_{\vec{u}}(-1, 4, 2)$.

Solution: We will need the gradient of h. Computing this, we get

$$\nabla h = \frac{3x^2 y}{1+z^2}\vec{\imath} + \frac{x^3}{1+z^2}\vec{\jmath} - \frac{2x^3 yz}{(1+z^2)^2}\vec{k}.$$

We evaluate ∇h at the point $(-1, 4, 2)$, getting

$$\nabla h(-1,4,2) = \frac{12}{5}\vec{\imath} - \frac{1}{5}\vec{\jmath} + \frac{16}{25}\vec{k}.$$

Since $\vec{u}$ is already a unit vector, we find immediately that

$$\begin{aligned} h_{\vec{u}}(-1,4,2) &= (\nabla h(-1,4,2)) \cdot \vec{u} \\ &= \frac{1}{\sqrt{17}}\left(\frac{24}{5} + \frac{3}{5} - \frac{32}{25}\right) \\ &= \frac{103}{25\sqrt{17}}. \end{aligned}$$

(b) Find a unit vector pointing in a direction in which the rate of change in h at the point $(-1, 4, 2)$ is zero.

Solution: We need to find a vector perpendicular to

$$\nabla h(-1,4,2) = \frac{12}{5}\vec{\imath} - \frac{1}{5}\vec{\jmath} + \frac{16}{25}\vec{k}.$$

Suppose $\vec{w}$ is such a vector, and write $\vec{w} = w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}$. Then $\vec{w}$ must satisfy the condition $\vec{w} \cdot \nabla h(-1, 4, 2) = 0$, that is

$$\frac{12}{5}w_1 - \frac{1}{5}w_2 + \frac{16}{25}w_3 = 0.$$

There are infinitely many solutions to this single equation; one of them is $w_1 = 1$, $w_2 = 12$, $w_3 = 0$. That is, the vector

$$\vec{w} = \vec{\imath} + 12\vec{\jmath}$$

is perpendicular to $\nabla h(-1, 4, 2)$. We are asked for a unit vector, so we divide our vector $\vec{w}$ by its length, getting

$$\frac{1}{\sqrt{145}}(\vec{\imath} + 12\vec{\jmath}).$$