Math 203(01) **Hour Exam II Solution** **16 November 2001**

1. Find the linear Taylor approximation $L(x,y)$ and the quadratic Taylor approximation $Q(x,y)$ to the function f given by

$$f(x,y) = y^2 e^{-2x}$$

at the point $(0,3)$.

Solution: We'll need the first and second partials of f. We get

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -2y^2 e^{-2x}; & \left.\frac{\partial f}{\partial x}\right|_{(0,3)} &= -18 \\
\frac{\partial f}{\partial y} &= 2y e^{-2x}; & \left.\frac{\partial f}{\partial y}\right|_{(0,3)} &= 6 \\
\frac{\partial^2 f}{\partial x^2} &= 4y^2 e^{-2x}; & \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,3)} &= 36 \\
\frac{\partial^2 f}{\partial x \partial y} &= -4y e^{-2x}; & \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,3)} &= -12 \\
\frac{\partial^2 f}{\partial y^2} &= 2e^{-2x}; & \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,3)} &= 2.
\end{aligned}$$

We also need to know that $f(0,3) = 9$. For the linear approximation, we get

$$L(x,y) = 9 - 18x + 6(y-3).$$

For the quadratic approximation, we get

$$Q(x,y) = 9 - 18x + 6(y-3) + 18x^2 - 12x(y-3) + (y-3)^2.$$

2. (a) Find all the critical points of the function f given by

$$f(x,y) = x^2y + 2x^3 - 4y.$$

(Don't bother to classify them.)

Solution: The partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2xy + 6x^2\\ \frac{\partial f}{\partial y} &= x^2 - 4.\end{aligned}$$

Setting these equal to zero, we get the system

$$\begin{aligned}2xy + 6x^2 &= 0\\ x^2 - 4 &= 0.\end{aligned}$$

The second equation has the solutions $x = \pm 2$. If $x = 2$, then the first equation becomes

$$4y + 24 = 0$$

so $y = -6$. If $x = -2$, the second equation becomes

$$-4y + 24 = 0$$

and so $y = 6$. The critical points are $(2, -6)$ and $(-2, 6)$.

(b) The function $f(x,y) = x^2 + 2y^2 + x^2y$ has critical points at $(0,0)$, $(2,-1)$, and $(-2,-1)$. Classify each of these as a local minimum, a local maximum, or a saddle point.

Solution: We'll need the first and second partials. We have

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 2xy \\ \frac{\partial f}{\partial y} &= 4y + x^2 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2 + 2y \\ \frac{\partial^2 f}{\partial x \partial y} &= 2x \\ \frac{\partial^2 f}{\partial y^2} &= 4 \end{aligned}$$

so that

$$D = 4(2 + 2y) - 4x^2.$$

Evaluating D at the given points we get

$$\begin{aligned} D(0,0) &= 8 \\ D(\pm 2, -1) &= -16. \end{aligned}$$

We conclude that $(\pm 2, -1)$ are saddle points. To classify $(0,0)$ we need to evaluate either $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 f}{\partial y^2}$ at $(0,0)$. In either case, we get a positive number, so we conclude that $(0,0)$ is a local minimum.

3. Use the Lagrange multiplier method to find the absolute maximum and minimum values of $f(x, y) = xy$ on the region $x^2 + 4y^2 \leq 8$.

Solution: We have

$$\nabla f = y\vec{\imath} + x\vec{\jmath},$$

from which we determine that $(0, 0)$ is a critical point for f. Next we set $g(x, y) = x^2 + 4y^2$

and find

$$\nabla g = 2x\vec{\imath} + 8y\vec{\jmath}.$$

Setting $\nabla f = \lambda \nabla g$ and $g(x, y) = 8$, we get the system of equations

$$\begin{aligned} y &= 2\lambda x \\ x &= 8\lambda y \\ x^2 + 4y^2 &= 8. \end{aligned}$$

We'd like to solve the first two equations for λ, but in order to do so, we have to divide by x and by y. Before we do this, we have to be sure that $x \neq 0$ and $y \neq 0$ in the solution. If $x = 0$, then the first equation says $y = 0$ as well, but the point $(0, 0)$ doesn't satisfy the third equation. Thus $x \neq 0$ in the solution. By a similar argument, $y \neq 0$ in the solution. Thus we have

$$\lambda = \frac{y}{2x}\lambda = \frac{x}{8y}.$$

This implies that

$$\begin{aligned} \frac{y}{2x} &= \frac{x}{8y} \\ 8y^2 &= 2x^2 \end{aligned}$$

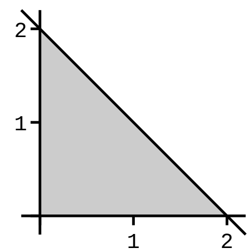
from which we get $x^2 = 4y^2$. Now from $x^2 = 4y^2$ and $x^2 + 4y^2 = 8$, we get $2x^2 = 8$, so $x^2 = 4$ so $x = \pm 2$. If $x = 2$, then $4y^2 = 4$ so $y = \pm 1$. Also, if $x = -2$, then $4y^2 = 4$, so $y = \pm 1$ again. There are four boundary points to check: $(\pm 2, \pm 1)$. We evaluate f at each of the five points we have identified. We get

$$\begin{aligned} f(0, 0) &= 0 \\ f(2, 1) &= 2 \\ f(2, -1) &= -2 \\ f(-2, 1) &= -2 \\ f(-2, -1) &= 2. \end{aligned}$$

The smallest of these numbers is -2 and the largest is 2.

4. Find the volume lying above the triangle (in the xy-plane) with vertices $(0,0)$, $(2,0)$, and $(0,2)$ and below the plane $z = 1 + x + y$.

Solution: We use a double integral of $1 + x + y$ over the given region.



The top boundary of the triangle is the line $y = 2 - x$, so the triangle can be described in rectangular coordinates as

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 2 \\ 0 & \leq & y & \leq & 2-x \end{array}$$

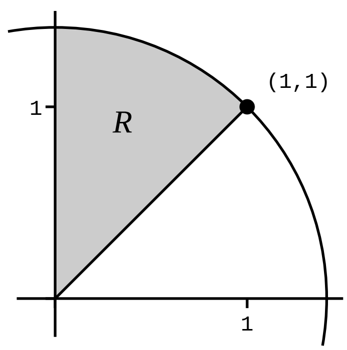
The integral is thus

$$\begin{aligned} \int_0^2 \int_0^{2-x} 1 + x + y \, dy \, dx &= \int_0^2 \left[y + xy + \frac{y^2}{2} \right]_0^{2-x} dx \\ &= \int_0^2 (2-x) + x(2-x) + \frac{1}{2}(2-x)^2 \, dx \\ &= \int_0^2 4 - x - \frac{1}{2}x^2 \, dx \\ &= \left[4x - \frac{x^2}{2} - \frac{x^3}{6} \right]_0^2 \\ &= \frac{14}{3}. \end{aligned}$$

5. Let $f(x, y) = x^2$. Set up, *but do not evaluate*, the integral

$$\iint\limits_R f\,dA$$

where R is the region in the picture below. (Assume that the arc in the picture is circular.)



Do this in three ways

(a) In rectangular coordinates, as an integral $dy\,dx$.
Solution: The top of the region is on the circle $x^2 + y^2 = 2$, and the bottom is on the line $y = x$. We get

$$\iint\limits_R f\,dA \quad = \quad \int_0^1 \int_x^{\sqrt{2-x^2}} x^2\,dy\,dx.$$

(b) In rectangular coordinates, as an integral $dx\,dy$.
Solution: Between $y = 0$ and $y = 1$, the line $x = y$ serves as the right-hand boundary; between $y = 1$ and $y = \sqrt{2}$, the circle is the right-hand boundary. We get

$$\iint\limits_R f\,dA \quad = \quad \int_0^1 \int_0^y x^2\,dx\,dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} x^2\,dx\,dy.$$

(c) In polar coordinates.
Solution: This is the easy one. We have

$$\iint\limits_R f\,dA \quad = \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^3 \cos^2\theta\,dr\,d\theta.$$

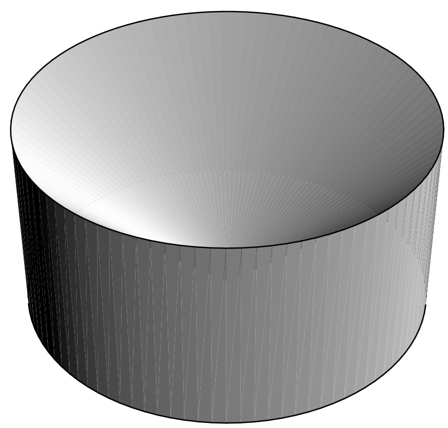
6. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Let W be the solid region lying above the plane $z=0$, below the cone $z = \sqrt{x^2+y^2}$, and inside the cylinder $x^2+y^2=4$. Set up, *but do not evaluate*, the integral

$$\iiint_W f\,dV$$

in two ways:

(a) In cylindrical coordinates.

Solution: The region W looks like a solid cylinder with a solid cone drilled out of the top. Here's a sketch:



We have $0 \le z \le r$ and $0 \le r \le 2$. The variable θ goes "all the way around." We also need to know that $x^2+y^2=r^2$, so that f can be expressed as $\sqrt{r^2+z^2}$. We get

$$\int_0^{2\pi}\int_0^2\int_0^r (\sqrt{r^2+z^2})r\,dz\,dr\,d\theta.$$

(b) In spherical coordinates.

Solution: We clearly have $0 \le \theta \le 2\pi$ and $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$. The limits on ρ are tricky, because ρ goes from 0 to the place where $x^2 + y^2 = 4$. We recall that

$$\sqrt{x^2 + y^2} \quad = \quad \rho \sin \varphi$$

which tells us that the integration limit $x^2 + y^2 = 4$ (that is, $\sqrt{x^2 + y^2} = 2$), corresponds to $\rho \sin \varphi = 2$, or

$$\rho = \frac{2}{\sin \varphi} = 2 \csc \varphi.$$

Also recalling that $\rho = \sqrt{x^2 + y^2 + z^2}$, we get

$$\iiint_W f\, dV \quad = \quad \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2 \csc \varphi} \rho^3 \sin \varphi \; d\rho \; d\varphi \; d\theta.$$