Math 203(01) **Practice Exam I** **5 October 2001**

1. Let $\vec{v} = 3\vec{\imath} - 4\vec{\jmath} + \vec{k}$ and $\vec{w} = \vec{\imath} - 2\vec{\jmath} + 2\vec{k}$.

   (a) Find the cosine of the angle θ between $\vec{v}$ and $\vec{w}$.
   Solution: We have

$$\begin{aligned} ||\vec{v}|| &= \sqrt{3^2 + (-4)^2 + 1^2} \\ &= \sqrt{26} \\ ||\vec{w}|| &= \sqrt{1^2 + (-2)^2 + 2^2} \\ &= 3 \\ \vec{v} \cdot \vec{w} &= 3 + 8 + 2 \\ &= 13. \end{aligned}$$

   Thus

$$\begin{aligned} \cos\theta &= \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \, ||\vec{w}||} \\ &= \frac{13}{3\sqrt{26}}. \end{aligned}$$

   (b) If $\vec{s} = 2\vec{\imath} + 3\vec{\jmath} + \alpha\vec{k}$, and $\vec{s}$ is perpendicular to $\vec{w}$, find the value of α.
   Solution: We need to find α to make $\vec{s} \cdot \vec{w} = 0$. That is, we need to solve

$$0 = 2 - 6 + 2\alpha.$$

   The solution is $\alpha = 2$.

2. Let A be the point $(1,2,1)$, B be the point $(3,2,5)$ and C be the point $(0,4,5)$.

   (a) Find an equation for the plane containing the points A, B, and C.
   Solution: We begin with two vectors

$$\begin{aligned} \overrightarrow{AB} &= 2\vec{\imath} - 4\vec{k} \\ \overrightarrow{AC} &= -\vec{\imath} + 2\vec{\jmath} + 4\vec{k}. \end{aligned}$$

   The cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ will give us a normal vector to the plane. We get

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 0 & 4 \\ -1 & 2 & 4 \end{vmatrix} \\ &= -8\vec{\imath} - 12\vec{\jmath} + 4\vec{k}. \end{aligned}$$

   We may as well simplify matters by taking a scalar multiple of this vector; let's say $2\vec{\imath} + 3\vec{\jmath} - \vec{k}$ is our normal. Now we just use one of the points on the plane (say A) to get the equation

$$2(x-1) + 3(y-2) - (z-1) = 0.$$

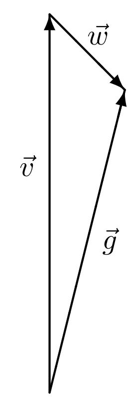
   (b) Find the area of triangle ABC.
   Solution: The area of the triangle is half the magnitude of the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$. We get

$$\begin{aligned} \text{Area} &= \frac{1}{2}\sqrt{(-8)^2 + (-12)^2 + 4^2} \\ &= 2\sqrt{14}. \end{aligned}$$

3. A pilot, flying at an altitude of 3000 feet, steers her airplane on a heading of due north at an airspeed of 100 knots. The ground track of the airplane is in a direction 6° east of north, and the airplane's ground speed is 80 knots. How fast is the wind blowing at 3000 feet?

(Give the answer both in exact form and as a decimal approximation.)

Solution:
Let $\vec{g}$ be the velocity of the airplane with respect to the ground, $\vec{v}$ be the velocity of the airplane with respect to the air, and $\vec{w}$ be the velocity of the wind. Then $\vec{v}+\vec{w}=\vec{g}$. Suppose the direction $\vec{\imath}$ is east and $\vec{\jmath}$ is north. We have



$$\begin{aligned}\vec{v} &= 100\vec{\jmath}\\ \vec{g} &= (80\sin 6^\circ)\vec{\imath}+(80\cos 6^\circ)\vec{\jmath}.\end{aligned}$$

From this, we easily find that

$$\begin{aligned}\vec{w} &= \vec{g}-\vec{v}\\ &= (80\sin 6^\circ)\vec{\imath}+(100-80\cos 6^\circ)\vec{\jmath}.\end{aligned}$$

The wind speed is given by

$$\begin{aligned}||\vec{w}|| &= \sqrt{80^2\sin^2 6^\circ+(100-80\cos 6^\circ)^2}\\ &= \sqrt{80^2(\sin^2 6^\circ+\cos^2 6^\circ)+100^2-16000\cos 6^\circ}\\ &= \sqrt{80^2+100^2-16000\cos 6^\circ}\\ &\approx 22 \text{ knots.}\end{aligned}$$

4. Let $\vec{w} = 4\vec{\imath} - \vec{\jmath} + 3\vec{k}$.

(a) Suppose $\vec{w} = \vec{w}_{\text{parallel}} + \vec{w}_{\text{perp}}$, where $\vec{w}_{\text{parallel}}$ is parallel to the vector $\vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{w}_{\text{perp}}$ is perpendicular to the vector $\vec{\imath} + \vec{\jmath} + \vec{k}$.
Write $\vec{w}_{\text{parallel}}$ and $\vec{w}_{\text{perp}}$ in components.
Solution: We need a unit vector $\vec{u}$ in the direction of $\vec{\imath} + \vec{\jmath} + \vec{k}$; we get

$$\vec{u} = \frac{1}{\sqrt{3}}(\vec{\imath} + \vec{\jmath} + \vec{k}).$$

The length of $\vec{w}_{\text{parallel}}$ is then given by

$$\begin{aligned}(\vec{w} \cdot \vec{u}) &= \frac{1}{\sqrt{3}}(4 - 1 + 3) \\ &= \frac{6}{\sqrt{3}} \\ &= 2\sqrt{3}.\end{aligned}$$

Thus

$$\begin{aligned}\vec{w}_{\text{parallel}} &= \frac{2\sqrt{3}}{\sqrt{3}}(\vec{\imath} + \vec{\jmath} + \vec{k}) \\ &= 2\vec{\imath} + 2\vec{\jmath} + 2\vec{k}\end{aligned}$$

and

$$\begin{aligned}\vec{w}_{\text{perp}} &= \vec{w} - \vec{w}_{\text{parallel}} \\ &= (4-2)\vec{\imath} + (-1-2)\vec{\jmath} + (3-2)\vec{k} \\ &= 2\vec{\imath} - 3\vec{\jmath} + \vec{k}\end{aligned}$$

(b) What is the distance from the point $(4, -1, 3)$ to the line $x = y = z$?
Solution: This distance happens to be exactly the length of $\vec{w}_{\text{perp}}$. We get

$$\begin{aligned}||\vec{w}_{\text{perp}}|| &= \sqrt{2^2 + (-3)^2 + 1^2} \\ &= \sqrt{14}\end{aligned}$$

5. Let $f(x, y) = \dfrac{y + \cos y}{1 + x^2 + y^2}$.

(a) Find the local linearization of f at $(2, 0)$.

Solution: We have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{-2x(y + \cos y)}{(1 + x^2 + y^2)^2} \\
\frac{\partial f}{\partial x}(2, 0) &= -\frac{4}{25} \\
\frac{\partial f}{\partial y} &= \frac{(1 + x^2 + y^2)(1 - \sin y) - 2y(y + \cos y)}{(1 + x^2 + y^2)^2} \\
\frac{\partial f}{\partial y}(2, 0) &= \frac{1}{5}.
\end{aligned}$$

Since $f(2, 0) = \dfrac{1}{5}$, the local linearization at $(2, 0)$ is given by

$$L(x, y) = \frac{1}{5} - \frac{4}{25}(x - 2) + \frac{1}{5}y.$$

(b) Suppose that x increases from 2 to 2.1. By what amount would y have to change (from 0) in order to keep $f(x, y)$ approximately constant?

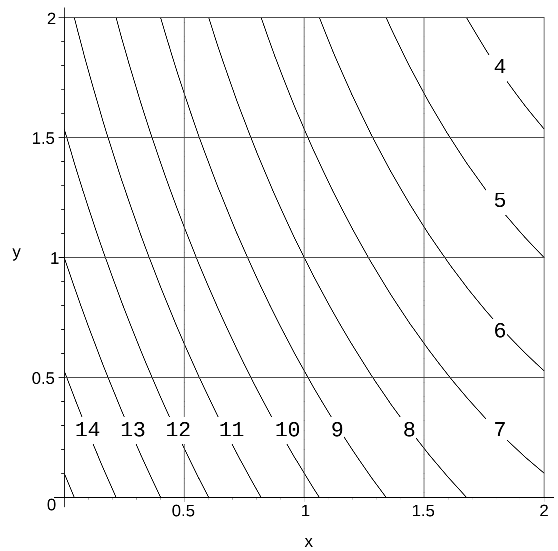
Solution: According to the local linearization, an increase of 0.1 in the value of x will decrease $f(x, y)$ by about

$$0.1 \times \frac{4}{25} = \frac{4}{250}.$$

We want to offset this decrease with an increase in y. According to the local linearization, near $(2, 0)$, the value of $f(x, y)$ increases about $\dfrac{1}{5}$ as fast as the value of y, so we need to increase y by about

$$5 \times \frac{4}{250} = \frac{4}{50}.$$

6. Here is a contour plot of a function $g(x, y)$.



Using the contour plot to approximate whatever values you need, construct an approximate local linearization for g at the point $(1, 1)$.

Solution: From the contour plot, we have $g(1, 1) = 0$. Moving up the line $x = 1$, it appears that g decreases about 1 unit over a change of 0.5 units in y; moving down the line $x = 1$, it appears that g increases about 1 unit for a change of -0.5 units in y. We estimate

$$\frac{\partial g}{\partial y}(1, 1) = -2.$$

Moving to the right along the line $y = 1$, it appears that g decreases by a little less than 2 units as x increases by 0.5 units; moving to the left, it appears that g increases by a little more than 2 units as x decreases by 0.5 units. We estimate

$$\frac{\partial g}{\partial x}(1, 1) = -4.$$

The local linearization of g at $(1, 1)$ is given by

$$L(x, y) = 8 - 4(x - 1) - 2(y - 1)$$

7. Let $f(x, y, z) = x^2 - y^2 + \dfrac{z^2}{4}$.

(a) Sketch the level surface $f(x, y, z) = 6$. Identify all the points where the surface intersects the coordinate axes.

Solution: We are looking for the surface whose equation is

$$x^2 - y^2 + \frac{z^2}{4} \quad = \quad 6.$$

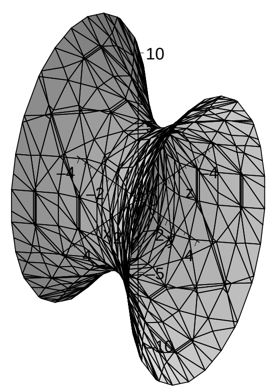
We rewrite this equation as

$$x^2 + \frac{z^2}{4} \quad = \quad y^2 + 6.$$

This says that the sections parallel to the xz-plane are ellipses, with longer axes in the z direction than in the x direction. The section at $y = 0$ is an ellipse with semi-axis length $\sqrt{6}$ along the x-axis, and seim-axis length $2\sqrt{6}$ along the z axis. We have an elliptic hyperboloid of one sheet about the y-axis, with intercepts at

$$(\pm\sqrt{6}, 0, 0) \quad \text{and} \quad (0, 0, \pm 2\sqrt{6}).$$

Here is a sketch:



(b) Find an equation for the plane tangent to the surface $f(x, y, z) = 6$ at the point $(3, 2, 2)$.

Solution: We have $\nabla f = 2x\vec{\imath} - 2y\vec{\jmath} + \dfrac{z}{2}\vec{k}$, and we get a normal vector to the tangent plane by evaluating

$$\nabla f(3, 2, 2) \quad = \quad 6\vec{\imath} - 4\vec{\jmath} - \vec{k}.$$

Using this vector and the point $(3, 2, 2)$, we can write an equation for the plane as

$$6(x - 3) - 4(y - 2) + (z - 2) \quad = \quad 0.$$

8. Let $g(x,y) = x^2y - x\ln y$.

(a) Let $\vec{u} = \dfrac{1}{13}(5\vec{\imath} - 12\vec{\jmath})$. Find $g_{\vec{u}}(3,2)$.

Solution: We need

$$\nabla g \quad = \quad (2xy - \ln y)\vec{\imath} + \left(x^2 - \frac{x}{y}\right)\vec{\jmath}.$$

Thus

$$\nabla g(3,2) \quad = \quad (12 - \ln 2)\vec{\imath} + \frac{15}{2}\vec{\jmath}.$$

Since $\vec{u}$ is already a unit vector, we need immediately get that

$$\begin{aligned} g_{\vec{u}}(3,2) &= (\nabla g(3,2)) \cdot \vec{u} \\ &= \frac{1}{13}\left(5(12 - \ln 2) - 12\frac{15}{2}\right) \\ &= \frac{1}{13}(-30 - 5\ln 2). \end{aligned}$$

(b) Find a vector pointing in the direction in which g decreases most rapidly from the point $(3,2)$.

Solution: A vector in this direction is given by $-\nabla g(3,2)$, which is

$$-(12 - \ln 2)\vec{\imath} - \frac{15}{2}\vec{\jmath}.$$