Math 203(01) Practice Exam II Solution 9 November 2001

1. Let $f(x, y) = 2x^2 - xy$. If $x = t\cos t$ and $y = t\sin t$, find $\dfrac{df}{dt}$. (Be sure to write your answer as a function of t.)

   Solution: By the chain rule, we know that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

   In this problem, we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x - y \\ \frac{\partial f}{\partial y} &= -x\end{aligned}$$

   and

$$\begin{aligned}\frac{dx}{dt} &= -t\sin t + \cos t \\ \frac{dy}{dt} &= t\cos t + \sin t.\end{aligned}$$

   Thus

$$\begin{aligned}\frac{df}{dt} &= (4x - y)(-t\sin t + \cos t) + (-x)(t\cos t + \sin t) \\ &= (4t\cos t - t\sin t)(-t\sin t + \cos t) + (-t\cos t)(t\cos t + \sin t) \\ &= (4t - t^2)\cos^2 t + (-4t^2 - 2t)\cos t\sin t + t^2\sin^2 t.\end{aligned}$$

2. Find the linear Taylor approximation $L(x, y)$ and the quadratic Taylor approximation $Q(x, y)$ to the function f given by

$$f(x, y) = (4 - x^2)\cos y$$

at the point $(1, 0)$.

Solution: We'll need the first and second partials of f. We get

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -2x\cos y; & \left.\frac{\partial f}{\partial x}\right|_{(1,0)} &= -2 \\
\frac{\partial f}{\partial y} &= -(4 - x^2)\sin y; & \left.\frac{\partial f}{\partial y}\right|_{(1,0)} &= 0 \\
\frac{\partial^2 f}{\partial x^2} &= -2\cos y; & \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,0)} &= -2 \\
\frac{\partial^2 f}{\partial x \partial y} &= 2x\sin y; & \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,0)} &= 0 \\
\frac{\partial^2 f}{\partial y^2} &= -(4 - x^2)\cos y; & \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,0)} &= -3.
\end{aligned}$$

We also need to know that $f(1, 0) = 3$. For the linear approximation, we get

$$L(x, y) = 3 - 2(x - 1).$$

For the quadratic approximation, we get

$$Q(x, y) = 3 - 2(x - 1) - (x - 1)^2 - \frac{3}{2}y^2.$$

3. Find and classify all the critical points of the function f given by

$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

Solution: The partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 6x^2 + y^2 + 10x \\ \frac{\partial f}{\partial y} &= 2xy + 2y. \end{aligned}$$

Setting the partials equal to zero, we get the system

$$\begin{aligned} 6x^2 + y^2 + 10x &= 0 \\ 2y(x+1) &= 0. \end{aligned}$$

The second equation implies that either $y = 0$ or $x = -1$. If $y = 0$, then the first equation becomes

$$\begin{aligned} 0 &= 6x^2 + 10x \\ &= 2x(3x+5) \end{aligned}$$

so that either $x = 0$ or $x = -\frac{5}{3}$. If, on the other hand, $x = -1$, then the first equation becomes

$$y^2 - 4 = 0$$

so $y = \pm 2$. There are four critical points. They are

$$(0,0), \quad \left(-\frac{5}{3}, 0\right), \quad (-1,2), \text{ and } (-1,-2).$$

To classify these points, we apply the second-partials test. We have

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 12x + 10 \\ \frac{\partial^2 f}{\partial x \partial y} &= 2y \\ \frac{\partial^2 f}{\partial y^2} &= 2x + 2. \end{aligned}$$

Thus

$$D = (12x+10)(2x+2) - 4y^2.$$

Evaluating D at the critical points, we get

$$\begin{aligned} D(0,0) &= 20 \\ D\left(-\frac{5}{3},0\right) &= \frac{40}{3} \\ D(-1,\pm 2) &= -16. \end{aligned}$$

We conclude at once that f has saddle points at $(-1,\pm 2)$. To classify the other critical points, we evaluate

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial x^2}\right|_{0,0} &= 12 \\ \left.\frac{\partial^2 f}{\partial x^2}\right|_{(-\frac{5}{3},0)} &= -10. \end{aligned}$$

We conclude that f has a local maximum at $\left(-\frac{5}{3},0\right)$ and a local minimum at $(0,0)$.

4. Use the Lagrange multiplier method to find the absolute maximum and minimum values of $f(x, y) = x^2 - 2x + y^2 - 2y$ on the disk $x^2 + y^2 \leq 4$.

Solution: We have

$$\nabla f = (2x - 2)\vec{\imath} + (2y - 2)\vec{\jmath},$$

from which we determine that $(1, 1)$ is a critical point for f. Next we set $g(x, y) = x^2 + y^2$ and find

$$\nabla g = 2x\vec{\imath} + 2y\vec{\jmath}.$$

Setting $\nabla f = \lambda \nabla g$ and $g(x, y) = 4$, we get the system of equations

$$\begin{aligned} 2x - 2 &= 2\lambda x \\ 2y - 2 &= 2\lambda y \\ x^2 + y^2 &= 4. \end{aligned}$$

We'd like to solve the first two equations for λ, but in order to do so, we have to divide by x and by y. Before we do this, we have to be sure that $x \neq 0$ and $y \neq 0$ in the solution. If $x = 0$, then the first equation says $-2 = 0$, which is false, so we conclude that $x \neq 0$. Similarly, we can conclude that $y \neq 0$. Thus we have

$$\begin{aligned} \lambda &= \frac{x - 1}{x} \\ \lambda &= \frac{y - 1}{y}. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{x - 1}{x} &= \frac{y - 1}{y} \\ x(y - 1) &= y(x - 1) \\ xy - x &= yx - y \end{aligned}$$

from which we get $x = y$. Now from $x = y$ and $x^2 + y^2 = 4$, we get $2x^2 = 4$, so $x^2 = 2$ so $x = \pm\sqrt{2}$. The two possible extreme points on the boundary are

$$(-\sqrt{2}, -\sqrt{2}) \quad \text{and} \quad (\sqrt{2}, \sqrt{2}).$$

We evaluate f at each of the three points to find the minimum and maximum. We get

$$f(1, 1) = 1 - 2 + 1 - 2$$

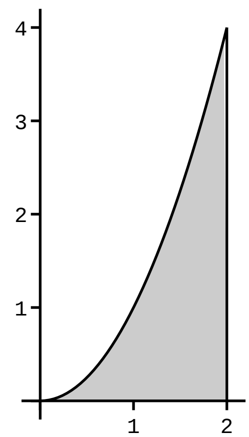
$$
\begin{aligned}
&= -2; \\
f(-\sqrt{2}, -\sqrt{2}) &= 2 + 2\sqrt{2} + 2 + 2\sqrt{2} \\
&= 4 + 4\sqrt{2}; \\
f(\sqrt{2}, \sqrt{2}) &= 2 - 2\sqrt{2} + 2 - 2\sqrt{2} \\
&= 4 - 4\sqrt{2}.
\end{aligned}
$$

The smallest of these numbers is -2 and the largest is $4 + 4\sqrt{2}$.

5. Let R be the region in the first quadrant of the xy-plane bounded by the x-axis, the line $x = 2$ and the parabola $y = x^2$. Find the volume of the solid region lying over R and under the surface $z = 4 + y - x^2$.

Solution: We'll set up a double integral. The region R may be described as

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 2 \\ 0 & \leq & y & \leq & x^2. \end{array}$$



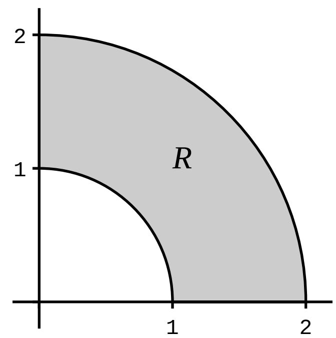
Thus the volume is given by

$$\begin{aligned} \int_0^2 \int_0^{x^2} 4 + y - x^2 \, dy \, dx &= \int_0^2 \left[4y + \frac{y^2}{2} - x^2 y \right]_0^{x^2} dx \\ &= \int_0^2 4x^2 - \frac{x^4}{2} \, dx \\ &= \left[\frac{4x^3}{3} - \frac{x^5}{10} \right]_0^2 \\ &= \frac{32}{3} - \frac{32}{10} \\ &= \frac{112}{15}. \end{aligned}$$

6. Let $f(x,y) = y$. Set up, *but do not evaluate*, the integral

$$\iint\limits_R f\,dA$$

where R is the region in the picture below. (Assume that the arcs in the picture are circular.)



Do this in two ways

(a) In rectangular coordinates.

Solution: Let's try putting dx on the outside. We'll have to split up the integral. The top of the region is always determined by the circle $x^2 + y^2 = 4$. The bottom, however, changes at $x = 1$. From $x = 0$ to $x = 1$, the bottom of the region is determined by the circle $x^2 + y^2 = 1$, but from $x = 1$ to $x = 2$, the bottom of the region is the line $y = 0$. We get

$$\iint\limits_R f\,dA = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} y\,dy\,dx.$$

(b) In polar coordinates.

Solution: Recalling that $y = r\sin\theta$, we get

$$\iint\limits_R f\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r^2 \sin\theta\,dr\,d\theta.$$

7. Let $f(x, y, z) = x^2 + y^2$. Let W be the solid region lying above the cone $z = \frac{1}{2}\sqrt{x^2 + y^2}$ and below the upper hemisphere $z = \sqrt{5 - x^2 - y^2}$. Set up, *but do not evaluate*, the integral

$$\iiint\limits_W f \, dV$$

in two ways:

(a) In cylindrical coordinates.

Solution: The bottom and the top of the region are given; we need only determine the shadow in the xy-plane. The sphere $x^2 + y^2 + z^2 = 5$ and the cone $z^2 = \frac{1}{4}(x^2 + y^2)$ intersect when

$$5 - x^2 - y^2 = \frac{1}{4}(x^2 + y^2)$$

that is, when $x^2 + y^2 = 4$. The shadow of W is a disk centered at the origin with radius 2. We get

$$\iiint\limits_W f \, dV = \int_0^{2\pi} \int_0^2 \int_{\frac{r}{2}}^{\sqrt{5-r^2}} r^3 \, dz \, dr \, d\theta.$$

(b) In spherical coordinates.

Solution: Since W is cut from a sphere by a cone (with constant φ), all the limits of integration will be constants. The only tricky part is finding the upper limit for φ. Using the fact that the slope of the cone is $\frac{1}{2}$, we can determine that the upper limit of φ satisfies $\tan\varphi = 2$. We also need to recall that $x^2 + y^2 = \rho^2 \sin^2\varphi$. We get

$$\iiint\limits_W f \, dV = \int_0^{2\pi} \int_0^{\tan^{-1}(2)} \int_0^{\sqrt{5}} \rho^4 \sin^3\varphi \, d\rho \, d\varphi \, d\theta.$$