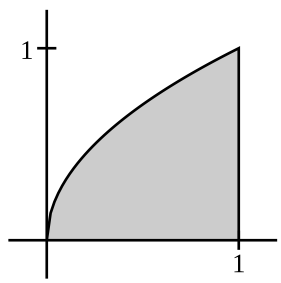
Practice Problems after Hour Exam 2

1. Compute

$$\int_0^1 \int_{y^2}^1 e^{x^{\frac{3}{2}}}\, dx\, dy.$$

Solution: As it stands, the integral is very nasty. We reverse the order of integration, and hope for the best. The region $0 \le y \le 1$, $y^2 \le x \le 1$ is pictured here.



The same region can be described as $0 \le x \le 1$, $0 \le y \le \sqrt{x}$. Setting up the integral this way, we get

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{x}} e^{x^{\frac{3}{2}}}\, dy\, dx &= \int_0^1 \sqrt{x} e^{x^{\frac{3}{2}}}\, dx \\ &= \left[\frac{2}{3} e^{x^{\frac{3}{2}}}\right]_0^1 \\ &= \frac{2}{3}(e-1). \end{aligned}$$

2. Let C be the helix centered on the z-axis, circling counterclockwise, making two full rotations while rising from $(4,0,0)$ to $(4,0,8)$.

   (a) Parametrize C.
   Solution: We get

$$\vec{r}(s) = 4\cos t\vec{\imath} + 4\sin t\vec{\jmath} + \frac{8t}{4\pi}\vec{k},$$

   with $0 \le t \le 4\pi$.

(b) Let $\vec{F} = -y\vec{\imath} + x\vec{\jmath} - z\vec{k}$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

Solution: We have

$$\vec{r}'(t) = -4\sin t\vec{\imath} + 4\cos t\vec{\jmath} + \frac{2}{\pi}\vec{k}$$

and

$$\vec{F}(\vec{r}) = -4\sin t\vec{\imath} + 4\cos t\vec{\jmath} - \frac{2t}{\pi}\vec{k}$$

so

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{4\pi} 16 - \frac{4t}{\pi^2}\, dt \\ &= 64\pi - \left[\frac{2t^2}{\pi^2}\right]_0^{4\pi} \\ &= 64\pi - 32. \end{aligned}$$

(c) Let $\vec{G} = 2xy^2z^2\vec{\imath} + 2yx^2z^2\vec{\jmath} + (2zx^2y^2 + 1)\vec{k}$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Solution: Looks like $\vec{F}$ is a gradient field. In fact, the potential function is $f(x, y, z) = x^2y^2z^2 + z$, so we get

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f(4, 0, 8) - f(4, 0, 0) \\ &= 8. \end{aligned}$$

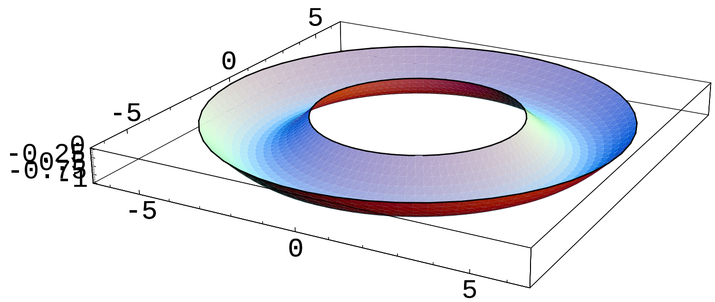
3. Let S be the surface parametrized by

$$\vec{r}(s, t) = s\cos t\vec{\imath} + s\sin t\vec{\jmath} + \sin s\vec{k}$$

with $\pi \le s \le 2\pi$ and $0 \le t \le 2\pi$.

(a) Sketch S and describe the orientation given by its parametrization.

Solution: This is a surface of revolution formed by taking the part of the sine curve between $x = \pi$ and $x = 2\pi$ and revolving it about the z-axis. Here's the picture



The s direction is radial (outward) and the t direction is circumferential (counterclockwise from above), so taking the variables in alphabetical order, we conclude that $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ points upward. The surface is oriented upward.

(b) Set up, but do not evaluate, an integral for the surface area of S.

Solution: Using the given parametrization, we have

$$\begin{aligned} \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} &= (\cos t\vec{\imath} + \sin t\vec{\jmath} + \cos s\vec{k}) \times (-s\sin t\vec{\imath} + s\cos t\vec{\jmath} + 0\vec{k}) \\ &= -s\cos s\cos t\vec{\imath} - s\cos s\sin t\vec{\jmath} + s\vec{k}. \end{aligned}$$

Thus

$$\begin{aligned} \left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right| &= \sqrt{s^2\cos^2 s + s^2} \\ &= s\sqrt{1+\cos^2 s}. \end{aligned}$$

The area of the surface is given by

$$\int_0^{2\pi}\int_{\pi}^{2\pi} s\sqrt{1+\cos^2 s}\, ds\, dt.$$

(c) Let $\vec{F} = y\vec{\imath} - x\vec{\jmath} + \vec{k}$. Calculate the flux of $\vec{F}$ upward through S.

Solution: From the previous part, our area vector is

$$-s\cos s\cos t\vec{\imath} - s\cos s\sin t\vec{\jmath} + s\vec{k},$$

which happens to be oriented upward, so we leave it alone. Using the parametrization and the given vector field, we get

$$\vec{F}(\vec{r}) \quad = \quad s\sin t\vec{\imath} - s\cos t\vec{\jmath} + \vec{k}.$$

Thus

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{A} &= \int_0^{2\pi} \int_\pi^{2\pi} -s^2 \sin t \cos t \cos s + s^2 \sin t \cos t \cos s + s \, ds \, dt \\
&= 2\pi \int_\pi^{2\pi} s \, ds \\
&= 3\pi^3.
\end{aligned}$$

4. Let C_1 be the parabolic curve $z = y^2$, $x = 0$, $-2 \leq y \leq 2$, oriented in the positive y direction. Let C_2 be the line segment from $(0, 2, 4)$ to $(0, -2, 4)$. Let C be the union of C_1 and C_2, so that C is a closed curve.

   Let $\vec{F} = \cos(x)\vec{\imath} + 3z\vec{\jmath} - y\vec{k}$.

   (a) Compute $\oint_C \vec{F} \cdot d\vec{r}$ by parametrizing C_1 and C_2. (You will have two integrals.)

   Solution: We parametrize C_1 as

$$\vec{r}_1(t) = 0\vec{\imath} + t\vec{\jmath} + t^2\vec{k}$$

with $-2 \leq t \leq 2$. Thus

$$\vec{F}(\vec{r}_1(t)) = 1\vec{\imath} + 3t^2\vec{\jmath} - t\vec{k}$$

and

$$\vec{r}_1'(t) = 0\vec{\imath} + \vec{\jmath} + 2t\vec{k}.$$

We get

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_{-2}^{2} 3t^2 - 2t^2 \, dt \\
&= \left[\frac{t^3}{3}\right]_{-2}^{2} \\
&= \frac{16}{3}.
\end{aligned}$$

We parametrize C_2 as

$$\vec{r}_2(t) = 0\vec{\imath} + (2 - t)\vec{\jmath} + 4\vec{k}$$

with $0 \le t \le 4$. We get

$$\vec{F}(\vec{r}_2(t)) = 1\vec{\imath} + 12\vec{\jmath} + (t-2)\vec{k}$$

and

$$\vec{r}_2'(t) = 0\vec{\imath} - \vec{\jmath} + 0\vec{\jmath}$$

so that

$$\begin{aligned} \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^4 (-12)\, dt \\ &= -48. \end{aligned}$$

We get

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \frac{16}{3} - 48 \\ &= -\frac{128}{3}. \end{aligned}$$

(b) Compute curl $\vec{F}$ and use Stokes's theorem to verify your answer to part (4a).
Solution: We have

$$\begin{aligned} \text{curl } \vec{F} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos x & 3z & -y \end{vmatrix} \\ &= -4\vec{\imath} \end{aligned}$$

The curve C bounds a surface S, described by $x = 0$, $-2 \le y \le 2$, $y^2 \le z \le 4$, oriented in the positive x direction. Since $(\text{curl } \vec{F}) \cdot \vec{\imath}$ is constantly equal to -4, we get

$$\begin{aligned} \iint_S \text{curl } \vec{F} \cdot d\vec{A} &= -4 \cdot (\text{area of } S) \\ &= -4 \int_{-2}^{2} (4 - y^2)\, dy \\ &= -4 \left[4y - \frac{y^3}{3} \right]_{-2}^{2} \\ &= -8 \left(\frac{8 - (-8)}{3} \right) \\ &= -\frac{128}{3}. \end{aligned}$$

5. Let W be the solid region $x^2+y^2 \le 4$ and $0 \le z \le 5$. Let

$$\vec{F}(x,y,z) \;=\; (2x+y\cos(z))\vec{\imath}+(3y+x\sin(z))\vec{\jmath}+z(5-z)\vec{k}.$$

(a) Explain why the flux of $\vec{F}$ through the top and bottom faces of W is zero.

Solution: The area vectors for the top and bottom faces are oriented upward and downward, respectively, so they are multiples of $\vec{k}$. On the top surface, $z=5$, and so the $\vec{k}$ component of $\vec{F}$ is zero. Thus there is no flux through the top surface. Similarly, on the bottom surface, $z=0$, so that $\vec{k}$ component of $\vec{F}$ is zero, and there is no flux through the bottom surface.

(b) Use the divergence theorem to compute the total flux of $\vec{F}$ through the boundary of W.

Solution: We have

$$\begin{aligned}\text{div } \vec{F} &= 2+3+5-2z \\ &= 10-2z\end{aligned}$$

Thus

$$\begin{aligned}\iiint_W \text{div } \vec{F}\,dV &= \int_0^{2\pi}\int_0^2\int_0^5 (10-2z)r\,dz\,dr\,d\theta \\ &= 2\pi\int_0^2 r\left[10z-z^2\right]_0^5 dr \\ &= 2\pi\int_0^2 25r\,dr \\ &= \left[25\pi r^2\right]_0^2 \\ &= 100\pi.\end{aligned}$$

(c) Determine the flux of $\vec{F}$ outward through the lateral surface of the boundary of W (that is, the part of the surface that isn't the top or the bottom).

Solution: The total flux is 100π, and none of it goes out through the top or bottom, so the flux through the lateral surface is 100π.