Math 311 Problem Set #10 Due 26 November 2002

For each integer $n \geq 3$, the dihedral group D_n is the group of symmetries of a regular n-gon. Suppose the vertices of the n-gon are numbered $1, 2, 3, \ldots, n$, going around counterclockwise. Consider the following two elements of D_n:

> The element r is a counterclockwise rotation of $2\pi/n$ radians;
> The element f is a reflection across the line determined by the center of the n-gon and the vertex numbered 1.

We already know that $r^n = e$ and $f^2 = e$ where e is the identity in D_n. It is not hard to show that r and f also satisfy the equality

$$rf = fr^{-1}. \tag{1}$$

(Draw a couple of diagrams if you want to convince yourself of this fact. Don't hand them in.) Furthermore, the set $\{r, f\}$ is a generating set for D_n. In fact, we have

$$D_n = \{e, r, r^2, \ldots, r^{n-1}, f, rf, r^2f, \ldots, r^{n-1}f\}.$$

Exercise: Determine the conjugacy classes in D_n. Here are the steps.

1. Use relation (1) to show that $r^k f = fr^{-k}$ for each integer $k \geq 0$.

2. Now look at the conjugacy relations in D_n. Use the result above and the usual rules of exponents to verify the following (k and m are non-negative integers):

(a) $(r^k)(r^m)(r^k)^{-1} = r^m$;
(b) $(r^k f)(r^m)(r^k f)^{-1} = r^{-m}$;
(c) $(r^k)(r^m f)(r^k)^{-1} = r^{2k+m} f$;
(d) $(r^k f)(r^m f)(r^k f)^{-1} = r^{2k-m} f$.

3. Using the r, f notation described above, list the conjugacy classes is D_6 and D_7. Does either group have a non-trivial center?

Make a conjecture about the conjugacy classes in D_n for any $n \geq 3$.

Cultural aside:

He thought he saw an Elephant
  That practised on a fife:
He looked again, and found it was
  A letter from his wife.
"At length I realise," he said,
  "The bitterness of Life!"

. . .

He thought he saw a Rattlesnake
  That questioned him in Greek:
He looked again, and found it was
  The Middle of Next Week.
"The one thing I regret," he said,
  "Is that it cannot speak!"

. . .

He thought he saw an Albatross
  That fluttered round the lamp:
He looked again, and found it was
  A Penny-Postage-Stamp.
"You'd best be getting home," he said:
  "The nights are very damp!"

He thought he saw a Garden-Door
  That opened with a key:
He looked again, and found it was
  A Double Rule of Three:
"And all its mystery," he said,
  "Is clear as day to me!"

He thought he saw an Argument
  That proved he was the Pope:
He looked again, and found it was
  A Bar of Mottled Soap.
"A fact so dread," he faintly said,
  "Extinguishes all hope!"

Lewis Carroll, from "The Mad Gardener's Song."