Math 311 Problem Set #2 Due 20 September 2002

Reading: Gallian, Chapter 3.

Exercises: Write your solutions in complete sentences.

1. (Chapter 3, exercise 4) Prove that in any group, an element and its inverse have the same order.

HINT: Start by proving that if g is an element of a group G with identity e, then for any positive integer n, $(g^{-1})^n = e$ if and only if $g^n = e$.

2. (Based on chapter 3, exercises 8 and 9) List the elements of $U(14)$ and determine the order of each element. Do the same for $U(20)$. Explain how this shows that $U(14)$ is a cyclic group and $U(20)$ is not.

3. (Chapter 3, exercise 15) For any group G, prove that $Z(G) = \bigcap_{a \in G} C(a)$.

Remark: To prove that two sets X and Y are equal, you do two things: first prove that every element of X is also an element of Y, then prove that every element of Y is also an element of X.

It's also important to realize that $x \in \bigcap_{a \in G} C(a)$ means that x is in $C(a)$ for every $a \in G$ Conversely, to show that $x \in \bigcap_{a \in G} C(a)$, you need to show that for an arbitrary $a \in G$, we have $x \in C(a)$.

Once you sort through all the definitions and quantifiers, this proof is fairly simple. The solution in the back of the book is a little too terse; do write out all the steps.

4. (Chapter 3, exercise 24) Suppose a belongs to a group and $|a| = 5$. Prove that $C(a) = C(a^3)$. Find an element a from some group such that $|a| = 6$ but $C(a) \neq C(a^3)$.

5. (Chapter 3, exercise 28) Consider the elements $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ from $SL(2, \mathbb{R})$. Find $|A|$, $|A|$, and $|AB|$.

Cultural aside:

And so it was only with the advent of pocket computers that the startling truth became finally apparent, and it was this:

Numbers written on restaurant checks within the confines of restaurants do not follow the same mathematical laws as numbers written on any other pieces of paper in any other parts of the Universe.

This single statement took the scientific world by storm. It completely revolutionized it. So many mathematical conferences got held in such good restaurants that many of the finest minds of a generation died of obesity and heart failure and the science of math was put back by years.

Slowly, however, the implications of the idea began to be understood. To begin with it had been too stark, too crazy, too much like what the man in the street would have said – "Oh, yes, I could have told you that." Then some phrases like "Interactive Subjectivity Frameworks" were invented, and everybody was able to relax and get on with it.

The small groups of monks who had taken up hanging around the major research institutes singing strange chants to the effect that the Universe was only a figment of its own imagination were eventually given a street theater grant and went away.

Douglas Adams, *The Restaurant at the End of the Universe*