

**Reading:** Gallian, Chapter 7.

**Exercises:** Write your solutions in complete sentences.

1. (a) Let  $K = \{0, 3\}$  be a subgroup of  $\mathbb{Z}_6$ . List the left cosets of  $K$  in  $\mathbb{Z}_6$ . List the right cosets of  $K$  in  $\mathbb{Z}_6$ .  
(b) Let  $H = \{e, (1\ 2)\}$ . List the left cosets of  $H$  in  $S_3$ . List the right cosets of  $H$  in  $S_3$ .
2. Let  $G$  be a group and  $H$  a subgroup of  $G$ . Prove that if  $G$  is Abelian, then  $aH = Ha$  for every  $a \in G$ . Find an example showing that the conclusion need not hold if  $G$  is non-Abelian.  
(You can use Part 6 of the “Properties of Cosets” Lemma to do this, but it’s probably easier just to show directly that  $aH \subseteq Ha$  and  $Ha \subseteq aH$  for every  $a \in G$ .)
3. (Chapter 7, exercise 10) Let  $a$  and  $b$  be non-identity elements of different orders in a group  $G$  of order 155. Suppose  $H$  is a subgroup of  $G$  and  $H$  contains both  $a$  and  $b$ . Prove that  $H = G$ .
4. Let  $H < A_5$  and suppose  $|A_5 : H| = 2$ .
  - (a) Prove that  $H$  contains every 3-cycle in  $A_5$ .
  - (b) Prove that  $H$  contains every 5-cycle in  $A_5$ .
  - (c) Use the results of parts 4a and 4b to prove that  $A_5$  contains no subgroup of order 30.
5. Draw a Cayley digraph of  $A_4$  on the generating set  $\{a, b\}$  where  $a = (1\ 2\ 3)$  and  $b = (2\ 3\ 4)$ . Try to make your drawing look as symmetric as possible.
6. (Optional) Prove the following generalization of 4a and 4b: Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $x$  an element of  $G$  with order  $p$ , where  $p$  is prime. If  $|G : H| < p$ , then  $x \in H$ .

## Cultural aside:

◦ *Is there such a thing as an unforgivable sin? If so, what is it and how do I commit it?*

• The worst sin, the act that all races and religions agree is against the laws of nature, is to attempt to divide by zero. Even computers will stop, horrified, in their tracks and chastise any operator crass enough to attempt such a thing. Division by zero may leave an indelible stain on the soul, which can be erased only by a full confession to a qualified mathematician. Even then, this sin must be atoned for in a purgatory filled with unbalanced checkbooks and tedious exercises in long division. But that's the way it has to be; otherwise, the entire system of checks and balances would fly out the window.

◦ *Why can't you divide by zero?*

• I can and often do divide by zero, but only after I've made the necessary preparations. First of all, I fast for forty-eight hours, consuming during that time only mildly fluoridated water. Next I don my special Mylar/Teflon division-by-zero suit. Then I put on a digitally recorded compact disc of Gregorian chants and begin with dividing very small numbers by other very small numbers. As the numbers get smaller, the sparks begin to fly. If all goes well, I take a deep breath and divide a very small number by zero. There's a flash of light, a muffled roar, and when I come to, the lab is filled with smoke and the scent of burning Mylar. So you see, you can divide by zero if you really want to. But chances are you just don't want to badly enough.

Dr. Science, *The Official Dr. Science Big Book of Science, Simplified*