Antiderivative problems – Solutions

1. Find the most general antiderivative $F(x)$ for the function $f(x) = \sin(x) + \dfrac{x^2+1}{x^3}$ (assume $x > 0$).

   Solution: We write $f(x) = \sin(x) + \dfrac{1}{x} + x^{-3}$. An antiderivative of $\sin(x)$ is $-\cos(x)$; an antiderivative of $\dfrac{1}{x}$ is $\ln x$, and an antiderivative of x^{-3} is $\dfrac{x^{-2}}{-2}$. We have

   $$F(x) = -\cos(x) + \ln x - \frac{1}{2x^2} + C$$

   for the most general antiderivative of the given function.

2. Find the most general antiderivative $G(x)$ for the function $g(x) = \dfrac{x^2+2}{x^2+1}$.

   Solution: We notice that

   $$\frac{x^2+2}{x^2+1} = \frac{x^2+1}{x^2+1} + \frac{1}{x^2+1} = 1 + \frac{1}{x^2+1}$$

   Since $\dfrac{1}{x^2+1}$ is the derivative of $\tan^{-1} x$, we get

   $$G(x) = x + \tan^{-1} x + C$$

3. Find $f(t)$ given that $f''(t) = 12t$, $f(1) = 7$, and $f'(1) = 3$.

   Solution: From $f''(t) = 12t$, we get that $f'(t) = 6t^2 + C_1$. Using the fact that $f'(1) = 3$, we get

   $$\begin{aligned} 3 &= f'(1) \\ &= 6(1)^2 + C_1 \\ &= 6 + C_1 \end{aligned}$$

so that $C_1 = -3$. From $f'(t) = 6t^2 - 3$, we get $f(t) = 2t^3 - 3t + C_2$. Using the fact that $f(1) = 7$, we get

$$\begin{aligned} 7 &= f(1) \\ &= 2(1)^3 - 3(1) + C_2 \\ &= -1 + C_2 \end{aligned}$$

so that $C_2 = 8$ and $f(t) = 2t^3 - 3t + 8$.

4. Find $g(x)$ given $g''(x) = 2 - 8x^{-3}$, $g(1) = -2$, and $g(2) = 5$.

   Solution: From $g''(x) = 2 - 8x^{-3}$, we get

$$\begin{aligned} g'(x) &= 2x - \frac{8x^{-2}}{-2} + C_1 \\ &= 2x + 4x^{-2} + C_1 \end{aligned}$$

   Taking another antiderivative, we get

$$\begin{aligned} g(x) &= x^2 + \frac{4x^{-1}}{-1} + C_1x + C_2 \\ &= x^2 - \frac{4}{x} + C_1x + C_2 \end{aligned}$$

   Now the condition $g(1) = -2$ says that

$$-2 = 1 - 4 + C_1 + C_2$$

   so that $C_1 + C_2 = 1$. The condition $g(2) = 5$ says that

$$5 = 4 - 2 + 2C_1 + C_2$$

   so that $2C_1 + C_2 = 3$. We have the system

$$\begin{array}{rcrcl} C_1 & + & C_2 & = & 1 \\ 2C_1 & + & C_2 & = & 3 \end{array}$$

   The solution is $C_1 = 2$ and $C_2 = -1$, so we have

$$g(x) = x^2 - \frac{4}{x} + 2x - 1$$