Math 101(02) Solution
Hour Exam I 23 September 2003

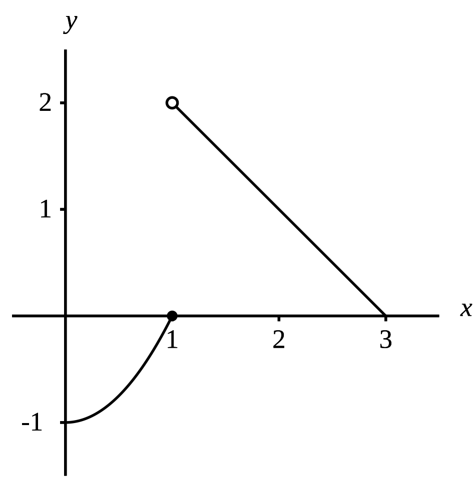
---

1. Let f be the function with domain $[0, 3]$ given by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } 0 \le x \le 1 \\ 3 - x & \text{if } 1 < x \le 3 \end{cases}$$

   (a) Sketch a graph of f on one of the grids provided. (The other grid is for practice; be sure to indicate which one you want to count as your answer.)

   Solution:



   (b) Find $f \circ f(3)$.

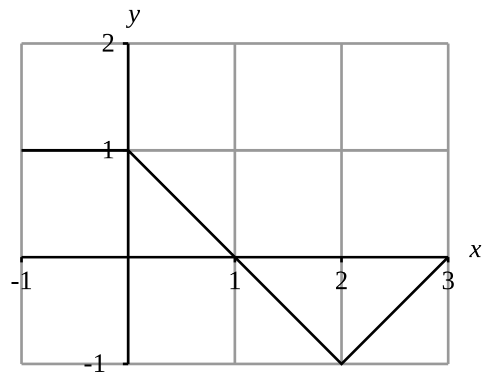
   We have $f(f(3)) = f(0) = -1$.

   (c) Find $f^{-1}\left(\frac{1}{2}\right)$.

   The value of $f(x)$ is $\frac{1}{2}$ when $x = \frac{5}{2}$, so

$$f^{-1}\left(\frac{1}{2}\right) = \frac{5}{2}.$$

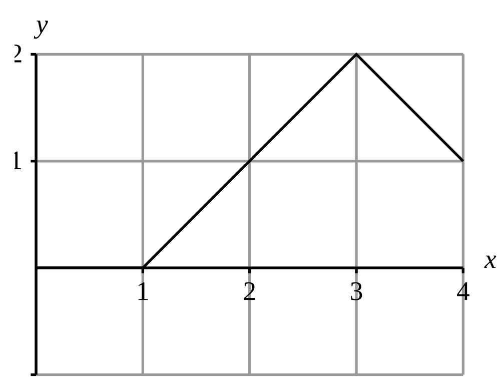
2. Here is the graph of a function f, defined on the interval $[-1, 3]$.



On one of the blank grids, draw a graph of the function g given by

$$g(x) \quad = \quad 1 - f(x-1).$$

Solution:



3. The operator of a railroad freight line finds that it requires 300 gallons of fuel to haul 2000 tons of freight from Atchison to Topeka, a distance of 50 miles. When the same train, travelling the same route, is loaded with 2600 tons of freight, it burns 375 gallons of fuel.

   (a) Assuming a linear model is appropriate, write g, the fuel comsumption, as a function of w, the weight of the load. Remember to specify units for g and w in your answer.

   Solution: We know $g = 375$ (in gallons) when $w = 2600$ (in tons) and $g = 350$ when $w = 2000$. The slope of the line is therefore

$$\begin{aligned} \frac{\text{rise}}{\text{run}} &= \frac{375 - 350}{2600 - 2000} \\ &= \frac{75}{600} \\ &= \frac{1}{8}. \end{aligned}$$

   We can write the equation using the point-slope formula and the point $(2000, 300)$. We get

$$g - 300 = \frac{1}{8}(w - 2000).$$

   Solving this for g, we get

$$g = \frac{1}{8}w + 50.$$

   The required function is

$$g(w) = \frac{1}{8}w + 50$$

   where g is in gallons and w is in tons.

   (b) What is the value of the g-intercept in part 3a, and what is its significance?

   The g-intercept is 50. An empty train making the run from Atchison to Topeka would burn 50 gallons of fuel.

4. A deck built over the waters of the Bay of Fundy is four feet above the surface of the water at high tide and 20 feet above the surface of the water at low tide. Assuming that the tides have a period of 12.5 hours, use a sine curve to construct a mathematical model of the height h of the deck above the surface of the water as a function of time t.

   Remember to specify units for h and t.

   The mean sea level is $\dfrac{20+4}{2} = 12$ feet below the surface of the deck, and the amplitude of the tides is ± 8 feet. The period is given as 12.5 hours. Our model is

$$h(t) \quad = \quad 12 + 8\sin\left(\frac{2\pi t}{12.5}\right)$$

   where h is in feet and t is in hours.

5. Given that $f(x) = \dfrac{2x+5}{x-1}$ find a formula for $f^{-1}(x)$.

Solution: We write

$$x = \frac{2y+5}{y-1}$$

and solve for y. We get

$$\begin{aligned} xy - x &= 2y + 5 \\ xy - 2y &= 5 + x \\ y(x-2) &= 5 + x \\ y &= \frac{5+x}{x-2}. \end{aligned}$$

Thus $f^{-1}(x) = \dfrac{5+x}{x-2}$.

6. (a) Solve the equation $2^{x+5} = 3 \times 4^{x-1}$ for x.
Write your answer in terms of natural logs.
Solution: Taking natural logs of both sides, we get

$$\begin{aligned} (x+5)\ln 2 &= \ln 3 + (x-1)\ln 4 \\ (x+5)\ln 2 &= \ln 3 + (2x-2)\ln 2 \\ 7\ln 2 - \ln 3 &= x\ln 2 \\ x &= \frac{7\ln 2 - \ln 3}{\ln 2}. \end{aligned}$$

(b) Solve the equation $\log_2 x - \log_2(x-1) = 2$ for x.
Write your answer in exact form.
Solution: We have

$$\log_2 \frac{x}{x-1} = 4$$

from which we get

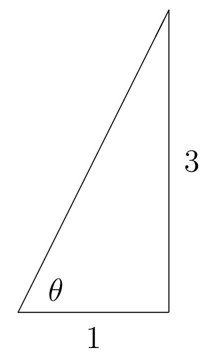
$$\begin{aligned} \frac{x}{x-1} &= 4 \\ x &= 4x - 4 \\ 4 &= 3x \end{aligned}$$

so $x = \dfrac{4}{3}$ is the answer.

7. (a) Find the exact value of $\cos(\tan^{-1} 3)$.

Let θ denote $\tan^{-1}(3)$. Then θ is the indicated angle in the right triangle at right.

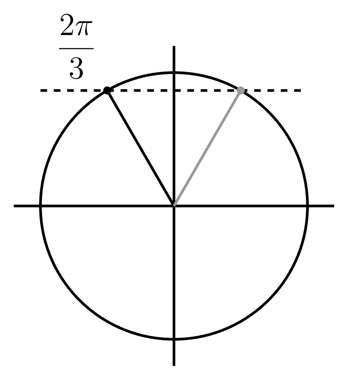
The hypotenuse of this triangle has length $\sqrt{9+1} = \sqrt{10}$, so the cosine of θ is $\dfrac{1}{\sqrt{10}}$.



(b) Find the exact value of $\sin^{-1}\left(\sin\left(\dfrac{2\pi}{3}\right)\right)$.

Give your answer in radians

Solution: In the unit circle picture at right, the darker of the two rays makes an angle of $\dfrac{2\pi}{3}$ with the positive x-axis, so it intersects the circle in the point $\left(\cos\left(\dfrac{2\pi}{3}\right), \sin\left(\dfrac{2\pi}{3}\right)\right)$.



The angle between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ that has the same sine is indicated by the lighter ray. Since the angle $\dfrac{\pi}{2}$ lies exactly half way between the two rays, the lighter ray must make an angle of

$$\frac{\pi}{2} - \left(\frac{2\pi}{3} - \frac{\pi}{2}\right) \quad = \quad \frac{\pi}{3}$$

with the positive x-axis. The angle we want is $\dfrac{\pi}{3}$.