Math 101(02)
Hour Exam II

Solution
24 October 2003

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1. Let f be the function on the interval $[-2, 3]$ given by

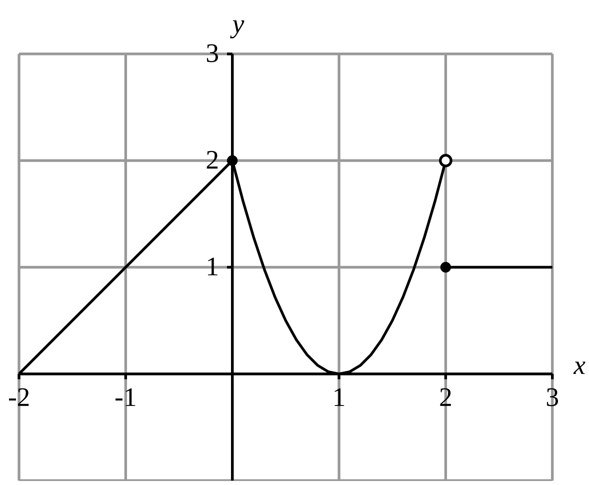
$$f(x) = \begin{cases} x+2 & \text{if } -2 \le x < 0 \\ 2(x-1)^2 & \text{if } 0 \le x < 2 \\ 1 & \text{if } 2 \le x < 3 \end{cases}$$

(a) Find $\lim_{x \to 0} f(x)$.

(b) Find $\lim_{x \to 2} f(x)$.

(c) Find all numbers x in $[-2, 3]$ at which f is not differentiable.

Solution: A graph of f will be helpful. Here it is:



(a) From the graph, it's clear that $\lim_{x \to 0^-} f(x) = 2$ and $\lim_{x \to 0^+} f(x) = 2$, so that $\lim_{x \to 0} f(x)$ is 2.

(b) The left- and right-hand limits do not agree at $x = 2$, so $\lim_{x \to 2} f(x)$ does not exist.

(c) The function f fails to be differentiable at $x = 2$, because it is discontinuous there. It also fails to be differentiable at $x = 0$, because the slope coming into $x = 0$ from the left is positive and the slope coming in from the right is negative. The graph has a corner at $x = 0$.

2. Compute the following limits algebraically. Show your work.

(a) Find $\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4}$

Solution: We have

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} &= \lim_{x\to 4} \frac{(2x+1)-9}{(x-4)(\sqrt{2x+1}+3)} \\
&= \lim_{x\to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} \\
&= \lim_{x\to 4} \frac{2}{\sqrt{2x+1}+3} \\
&= \frac{2}{\sqrt{9}+3} \\
&= \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

(b) Find $\lim_{x\to 0^+} \frac{x-3}{x^3-7x^2+12x}$

Solution: We have

$$\begin{aligned}
\lim_{x\to 0^+} \frac{x-3}{x^3-7x^2+12x} &= \lim_{x\to 0^+} \frac{x-3}{x(x-3)(x-4)} \\
&= \lim_{x\to 0^+} \frac{1}{x(x-4)}
\end{aligned}$$

Now we substitute 0^+ for x, getting $\frac{1}{0^+ \times -4^+}$. The numerator is positive and the denominator is a negative number. As the denominator approaches zero through negative values, the quotient tends to $-\infty$.

(c) Find $\lim_{x\to -\infty} \frac{2x-4}{\sqrt{x^2+1}}$.

Solution: As $x \to -\infty$, we have $x = -\sqrt{x^2}$. We divide the numerator of the given fraction by x and the denominator by $-\sqrt{x^2}$ to get

$$\lim_{x\to -\infty} \frac{2x-4}{\sqrt{x^2+1}} = \lim_{x\to -\infty} \frac{2-\frac{4}{x}}{-\sqrt{\frac{x^2+1}{x^2}}}$$

$$\begin{aligned} &= -\lim_{x\to-\infty} \frac{2-\frac{4}{x}}{\sqrt{1+\frac{1}{x^2}}} \\ &= -\frac{2}{\sqrt{1}} \\ &= -2 \end{aligned}$$

3. Use any appropriate differentiation rules to compute $f'(x)$. Do not simplify.

   (a) $f(x) = x^2 2^x$
   Solution: We have
   $$f'(x) \quad = \quad x^2 \cdot (2^x \ln 2) + 2x \cdot 2^x$$

   (b) $f(x) = \dfrac{x^5}{\sqrt{x^3+x}}$
   Solution: We use the quotient and chain rules to get
   $$f'(x) \quad = \quad \frac{\sqrt{x^3+x}\cdot 5x^4 - x^5 \cdot \frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+1)}{x^3+x}$$

   (c) $f(x) = \cos(xe^{-x})$
   Solution: We have
   $$f'(x) \quad = \quad -\sin(xe^{-x}) \cdot (e^{-x} - xe^{-x})$$

4. The amount of time it takes for a quart of skim milk to spoil depends on the temperature at which it is kept. Let $f(t)$ denote the number of hours before a quart of skim milk spoils if it is kept at temperature t, where t is measured in degrees Fahrenheit.

   Suppose that $f(40) = 120$ and $f'(40) = -5$.

   (a) What are the units of $f'(t)$? What is the meaning of the statement $f'(40) = -5$?
   Solution: The units of $f'(t)$ are hours per Fahrenheit degree. The statement $f'(40) = -5$ tells us that if the milk is being kept near 40 degrees Fahrenheit, a rise of one Fahrenheit degree will shorten its shelf life by about five hours.

   (b) What is a good estimate for $f(38)$?
   Solution: To get to 38 degrees Fahrenheit, we lower the temperature from 40 by two degrees. Each one-degree reduction in temperature (near 40 degrees) increases the shelf life of the milk by five hours. Thus $f(38)$ is about $120 + 10 = 130$ hours.

5. Let $f(x) = x^3 + 3x^2 - 23x + 5$. Find all points on the curve $y = f(x)$ where the tangent line has slope 1. (Remember to give both the x- and y-coordinates of each point.)

Solution: We have

$$f'(x) = 3x^2 + 6x - 23.$$

We want to find the points where $f'(x) = 1$, that is, where

$$\begin{aligned} 3x^2 + 6x - 23 &= 1 \\ 3x^2 + 6x - 24 &= 0 \\ 3(x^2 + 2x - 8) &= 0 \\ 3(x+4)(x-2) &= 0 \end{aligned}$$

The x-coordinates of the points are -4 and 2. The corresponding y-coordinates are

$$\begin{aligned} f(-4) &= -64 + 48 + 92 + 5 \\ &= 81 \\ f(2) &= 8 + 12 - 46 + 5 \\ &= -21 \end{aligned}$$

So the points are $(-4, 81)$ and $(2, -21)$.

6. Suppose f and g are functions and that $C(x) = f(g(x))$ and $P(x) = f(x)g(x)$. Suppose also that the following values of f, g, f', and g' are known.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	4	1	3
3	2	-1	3	4
4	-1	3	2	-2

(a) Find $C'(2)$.

Solution: By the chain rule, we get

$$\begin{aligned} C'(2) &= f'(g(2)) \cdot g'(2) \\ &= f'(4) \cdot 3 \\ &= 2 \cdot 3 \\ &= 6. \end{aligned}$$

(b) Find $P'(3)$.

Solution: By the product rule, we have

$$\begin{aligned} P'(3) &= f(3)g'(3) + f'(3)g(3) \\ &= 2 \cdot 4 + 3 \cdot (-1) \\ &= 5. \end{aligned}$$