Math 101(02) Solution
Hour Exam III 21 November 2003

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1. Find y'.

   (a) $y = \dfrac{1}{\tan^{-1}(x)}$
   Solution: We have

$$\begin{aligned} y' &= -\frac{1}{(\tan^{-1}(x))^2} \cdot \frac{1}{1+x^2} \\ &= -\frac{1}{(1+x^2)(\tan^{-1}(x))^2}. \end{aligned}$$

   (b) $y = \log_5(100 - x^2)$
   Solution: We have

$$\begin{aligned} y' &= \frac{1}{(100-x^2)(\ln 5)}(-2x) \\ &= \frac{-2x}{(100-x^2)(\ln 5)} \end{aligned}$$

   (c) $y = x^{2x}$ (Write your answer in terms of x.)
   Solution: We write

$$\ln y = 2x \ln x$$

   and differentiate to get

$$\begin{aligned} \frac{y'}{y} &= 2x \cdot \frac{1}{x} + 2\ln x \\ &= 2 + 2\ln x \end{aligned}$$

   so that

$$\begin{aligned} y' &= y(2 + 2\ln x) \\ &= x^{2x}(2 + 2\ln x) \end{aligned}$$

2. Find an equation for the line tangent to the curve $xy^3 - y^2 + 3x^2 = 7$ at the point $(1,2)$.

Solution: We differentiate implicitly to get

$$\begin{aligned} 3xy^2\frac{dy}{dx} + y^3 - 2y\frac{dy}{dx} + 6x &= 0 \\ y^3 + 6x &= (2y - 3xy^2)\frac{dy}{dx} \end{aligned}$$

so that $\frac{dy}{dx} = \frac{y^3 + 6x}{2y - 3xy^2}$. At the point $(1,2)$, this gives $\frac{dy}{dx} = \frac{14}{-8} = -\frac{7}{4}$. An equation for the tangent line is

$$y - 2 = -\frac{7}{4}(x - 1)$$

3. A particle moves along the x-axis with position function given by

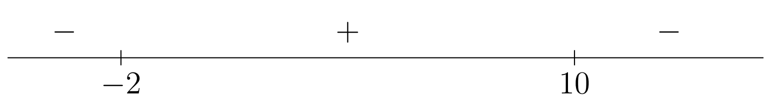
$$x(t) = 12t^2 - t^3 + 60t + 13$$

Find the time intervals in which the particle is both moving to the right and slowing down.

Solution: We need to determine the times at which the particle has positive velocity and negative acceleration. For the velocity, we have

$$\begin{aligned} x'(t) &= 24t - 3t^2 + 60 \\ &= -3(t^2 - 8t - 20) \\ &= -3(t - 10)(t + 2) \end{aligned}$$

The velocity changes sign (and thus the particle changes direction) when $x = -2$ and when $x = 10$. Here is a sign chart for $x'(t)$.



The particle is moving to the right for $-2 < t < 10$.

The particle's acceleration is given by $x''(t)$. We have

$$\begin{aligned} x''(t) &= 24 - 6t \\ &= 6(4 - t) \end{aligned}$$

This is clearly positive when $t < 4$ and negative when $t > 4$.

The time interval on which the particle has both positive velocity and negative acceleration is $4 < t < 10$.

4. Find the linearization $L(x)$ of the function $f(x) = \dfrac{1}{\sqrt[3]{x}}$ at $a = 8$ and use it to approximate $\dfrac{1}{\sqrt[3]{7.5}}$. (Write your approximation as a fraction.)

   Solution: We have $f(8) = \dfrac{1}{\sqrt[3]{8}} = \dfrac{1}{2}$, and $f'(x) = -\dfrac{1}{3}x^{-\frac{4}{3}}$ so that

$$\begin{aligned} f'(8) &= -\frac{1}{3}(8)^{-\frac{4}{3}} \\ &= -\frac{1}{48} \end{aligned}$$

   The linearization we want is given by

$$L(x) = \frac{1}{2} - \frac{1}{48}(x-8)$$

   We approximate $\dfrac{1}{\sqrt[3]{7.5}}$ as

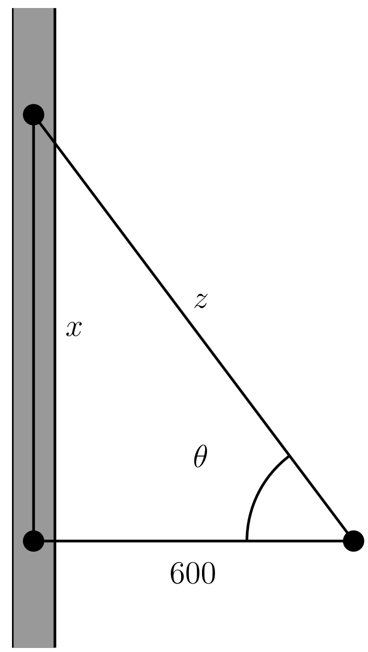
$$\begin{aligned} L(7.5) &= \frac{1}{2} - \frac{1}{48}(7.5-8) \\ &= \frac{1}{2} + \frac{1}{96} \\ &= \frac{49}{96} \end{aligned}$$

5. The vice-presidential motorcade travels north along Main Street, passing checkpoint alpha at exactly 12 noon. Geraldo has his television camera set up 600 feet due east of checkpoint alpha, and keeps the camera pointed toward the vice-president's limousine as it drives by at a steady speed of 50 feet per second.

   (a) How fast is the distance between Geraldo's camera and the vice-president's limousine changing when the limousine is 800 feet north of checkpoint alpha?

Solution: Let x be the distance between checkpoint alpha and the vice-president's limousine. Let z be the distance between Geraldo's camera and the limousine. From the picture, we see that

$$x^2 + 600^2 = z^2$$



We treat both x and z as functions of t and differentiate to get

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

We are told that $\frac{dx}{dt} = 50$. At the moment in question, we know that $x = 800$. We use the Pythagorean theorem to find that $z = 1000$ at the same moment. We get

$$2(800)(50) = 2(1000)\frac{dz}{dt}$$

We solve this for $\frac{dz}{dt}$. We get $\frac{dz}{dt} = 40$, so the distance is increasing at the rate of 40 feet per second.

(b) How fast is Geraldo's camera rotating at that same moment?

Solution: Let θ be the angle that Geraldo's camera makes with an east-west line through checkpoint alpha. From the same picture, we can conclude that $\tan\theta = \frac{x}{600}$. We treat both x and θ as functions of t and differentiate to get

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{600}\frac{dx}{dt}$$

At the moment in question, we know from the dimensions of the triangle that $\sec\theta = \frac{1000}{600} = \frac{5}{3}$. Using this along with the fact that $\frac{dx}{dt} = 50$, we get

$$\left(\frac{5}{3}\right)^2\frac{d\theta}{dt} = \frac{50}{600}$$

We solve this for $\frac{d\theta}{dt}$, getting $\frac{d\theta}{dt} = \frac{3}{100}$. The camera is rotating at the rate of

$\frac{3}{100}$ radians per second.

6. Find the following limits.

   (a) $\lim_{x\to 0} \frac{x^2}{\cos(2x) - 1}$

   Solution: The limit has the form $0/0$, which is indeterminate. We apply l'Hospital's rule to get

$$\lim_{x\to 0} \frac{x^2}{\cos(2x) - 1} = \lim_{x\to 0} \frac{2x}{-2\sin(2x)}$$

   The new limit also has form $0/0$, so we may apply l'Hospital again to get

$$\begin{aligned} \lim_{x\to 0} \frac{2x}{-2\sin(2x)} &= \lim_{x\to 0} \frac{2}{-4\cos(2x)} \\ &= -\frac{1}{2} \end{aligned}$$

   The limit is $-\frac{1}{2}$.

   (b) $\lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}}$

   Solution: This has the form 1^∞, which is indeterminate. We write

$$y = \lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}}$$

   and take logarithms to get

$$\ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin x)}{x}$$

   The new limit has the form $0/0$, so we may apply l'Hospital's rule to get

$$\begin{aligned} \lim_{x\to 0^+} \frac{\ln(1+\sin x)}{x} &= \lim_{x\to 0^+} \frac{\frac{\cos x}{1+\sin x}}{1} \\ &= \lim_{x\to 0^+} \frac{\cos x}{1+\sin x} \\ &= 1 \end{aligned}$$

   Since $\ln y = 1$, we conclude that $y = e$.

7. Find the absolute minimum and maximum values of $f(x) = x^3 - 9x$ on the interval $[-1, 2]$.

Solution: To find the critical numbers of the given function, we compute

$$\begin{aligned} f'(x) &= 3x^2 - 9 \\ &= 3(x^2 - 3) \end{aligned}$$

The derivative is zero at $x = \pm\sqrt{3}$, but only $\sqrt{3}$ lies in the domain. We evaluate f at this one critical point and at the endpoints of the interval. We get

$$f(-1) = 8; \quad f(\sqrt{3}) = -6\sqrt{3}; \quad f(2) = -10$$

The absolute minimum value is $f(\sqrt{3}) = -6\sqrt{3}$ and the absolute maximum value is $f(-1) = 8$.