Math 101(02) Solution
Final Exam December 2003

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1. Compute the derivatives $\frac{dy}{dx}$.

   (a) $y = x\sin^{-1}(x)$
   Solution:

   $$y' = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x)$$

   (b) $y = \frac{2x}{\sqrt{1-x^2}}$
   Solution:

   $$\begin{aligned} y' &= \frac{2\sqrt{1-x^2} + \frac{2x^2}{\sqrt{1-x^2}}}{1-x^2} \\ &= \frac{2(1-x^2)+2x^2}{(1-x^2)^{\frac{3}{2}}} \\ &= \frac{2}{(1-x^2)^{\frac{3}{2}}} \end{aligned}$$

   (c) $y = x^{3x-1}$
   Solution: To get started, we take logarithms of both sides to get

   $$\ln y = (3x-1)\ln x$$

   We differentiate to get

   $$\frac{1}{y}\frac{dy}{dx} = \frac{3x-1}{x} + 3\ln x$$

   and then solve for dy/dx, getting

   $$\frac{dy}{dx} = (3x-1)\ln x\left(\frac{3x-1}{x} + 3\ln x\right)$$

2. Compute the following limits.

(a) $\lim_{x\to 2^+} \dfrac{x^2-2x-3}{x^2-5x+6}$
We factor the numerator and denominator to get

$$\frac{x^2-2x-3}{x^2-5x+6} = \frac{(x-3)(x+1)}{(x-3)(x-2)}$$

from which it follows that

$$\lim_{x\to 2^+} \frac{x^2-2x-3}{x^2-5x+6} = \lim_{x\to 2^+} \frac{x+1}{x-2}$$

This limit has the form $\dfrac{3^+}{0^+}$, so we conclude that the limit is $+\infty$.

(b) $\lim_{x\to 0} \dfrac{e^{3x}-x-1}{\cos(x)-1}$
Solution: The limit has the indeterminate form 0/0, so we apply l'Hospital's rule, getting

$$\lim_{x\to 0} \frac{e^{3x}-x-1}{\cos(x)-1} = \lim_{x\to 0} \frac{3e^{3x}-1}{-\sin x}$$

The new limit has the form 2/0, so it is an infinite limit. Since

$$\lim_{x\to 0^+} \frac{3e^{3x}-1}{-\sin x} = -\infty$$

and

$$\lim_{x\to 0^-} \frac{3e^{3x}-1}{-\sin x} = +\infty$$

we conclude that $\lim_{x\to 0} \dfrac{3e^{3x}-1}{-\sin x}$ does not exist.

3. Let E be the ellipse with equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

(a) Find the slope of the line tangent to E at the point $\left(\dfrac{4\sqrt{5}}{3}, 2\right)$.

Solution: Using implicit differentiation (with respect to x), we get

$$\frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0$$

so that

$$\frac{dy}{dx} = -\frac{9x}{16y}$$

At the given point, we get

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{\left(\frac{4\sqrt{5}}{3},2\right)} &= -\frac{9 \times 4\sqrt{5}}{3 \times 16 \times 2} \\ &= -\frac{3\sqrt{5}}{8}\end{aligned}$$

(b) Find a point on E where the tangent line has slope -1.

Solution: We need to find a point (x, y) satisfying $-\frac{9x}{16y} = -1$ and $\frac{x^2}{16} + \frac{y^2}{9} = 1$. From the first equation, we get $y = \frac{9x}{16}$. We substitute this into the second equation to get

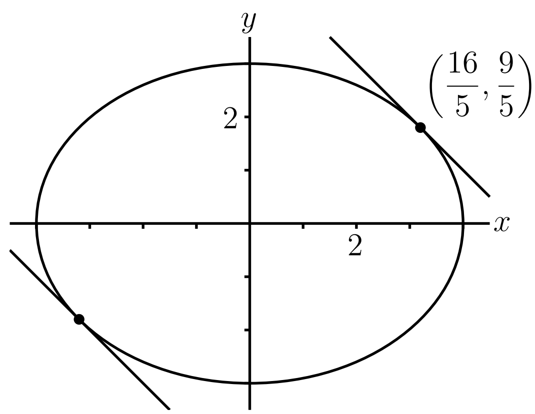
$$\frac{x^2}{16} + \frac{9x^2}{16^2} = 1$$

The common denominator is 16^2, and we get

$$\begin{aligned}\frac{16x^2 + 9x^2}{16^2} &= 1 \\ 25x^2 &= 16^2 \\ x &= \pm\frac{16}{5}\end{aligned}$$

There are two points on the ellipse where the tangent line has slope -1. They are

$$\left(\frac{16}{5}, \frac{9}{5}\right) \text{ and } \left(-\frac{16}{5}, -\frac{9}{5}\right)$$



4. Let $f(x) = \sqrt{x^3 + 3x}$.

(a) Find $L(x)$, the linearization of f at $a = 3$.
Solution: We have

$$f(3) = \sqrt{3^3 + 3 \times 3} = \sqrt{36} = 6$$

and

$$f'(x) = \frac{1}{2}(x^3 + 3x)^{-\frac{1}{2}}(3x^2 + 3)$$

so that

$$\begin{aligned} f'(3) &= \frac{1}{2}36^{-\frac{1}{2}}(30) \\ &= \frac{5}{2} \end{aligned}$$

The linearization is

$$L(x) = 6 + \frac{5}{2}(x - 3)$$

(b) Use your linearization to find an approximate solution to the equation

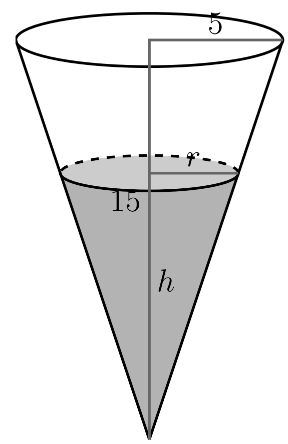
$$\sqrt{x^3 + 3x} = 6.2$$

Solution: We set $L(x)$ equal to 6.2 and solve. We get

$$\begin{aligned} 6 + \frac{5}{2}(x - 3) &= 6.2 \\ \frac{5}{2}(x - 3) &= \frac{1}{5} \\ x - 3 &= \frac{2}{25} \end{aligned}$$

so that $x = \dfrac{77}{25} = 3.08$ is a good estimate for the solution to the given equation.

5. A paper cup has the shape of a right circular cone with the point at the bottom. The cup's height is 15 cm and its radius is 5 cm. Water is leaking out through a hole in the bottom of the cup at the rate of 5 cm^3/s. How fast is the water level dropping when the water in the cup is 10 cm deep?

   Solution: Let V denote the volume of the water remaining in the cone. Let h and r denote the height and radius of the water remaining in the cone, as in the diagram at right.
   We are told that $dV/dt = -5$ and asked to determine dh/dt when $h = 10$. We recall that

   $$V = \frac{1}{3}\pi r^2 h$$

   so that

   $$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt} \qquad (1)$$

   From similar triangles, we get

   $$\frac{h}{r} = \frac{15}{5}$$

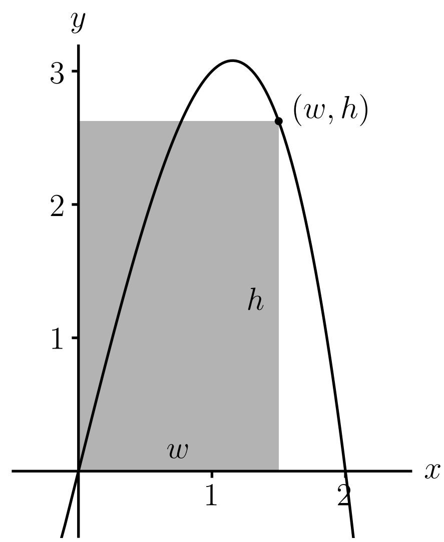
   so that $r = \dfrac{1}{3}h$ and thus $\dfrac{dr}{dt} = \dfrac{1}{3}\dfrac{dh}{dt}$. Also, when $h = 10$, we get $r = \dfrac{10}{3}$.
   Making all the substitutions into (1), we get

   $$\begin{aligned} -5 &= \frac{1}{3} \cdot \frac{100\pi}{9}\frac{dh}{dt} + \frac{2\pi}{3} \cdot \frac{100}{3} \cdot \frac{1}{3}\frac{dh}{dt} \\ &= \frac{300\pi}{27}\frac{dh}{dt} \end{aligned}$$

   so that $\dfrac{dh}{dt} = -5 \cdot \dfrac{27}{300\pi} = -\dfrac{9\pi}{20}$ centimeters per second.

6. Find the area of the largest rectangle with sides parallel to the coordinate axes, one corner at the origin, and the opposite corner in the first quadrant and on the curve $y = 4x - x^3$.

Solution: Let w and h denote the width and height of the rectangle whose area we want to maximize. The function we want to maxmimze is $A = wh$. To eliminate one of the variables, we use the fact that the point (w, h) lies on the graph $y = 4x - x^3$, so that $h = 4w - w^3$. We make this substitution to get



$$\begin{aligned} A(w) &= w(4w - w^3) \\ &= 4w^2 - w^4 \end{aligned}$$

From the picture, it's clear that to keep the point (w, h) in the first quadrant, we must have w in the interval $[0, 2]$. So look for critical numbers of A with w between 0 and 2. We get

$$\begin{aligned} A'(w) &= 8w - 4w^3 \\ &= 4w(2 - w^2) \end{aligned}$$

so that the critical numbers are $w = 0$ and $w = \pm 2$. Only the critical number $w = \sqrt{2}$ is of interest.

The absolute maximum value of A on $[0, 2]$ must occur at either $w = 0$, $w = \sqrt{2}$, or $w = 2$. We find that

$$A(0) = 0, \quad A(\sqrt{2}) = 4, \quad \text{and } A(2) = 0$$

so that the maximum value of A is 4. The largest rectangle satisfying the conditions in the problem has area 4.

7. Find the most general antiderivative $F(x)$ for the function f given by

$$f(x) = \frac{6x^4 + 2x - 12}{x^4}$$

Solution: We write

$$f(x) = 6 + 2x^{-3} - 12x^{-4}$$

and use the power rule to get

$$\begin{aligned} F(x) &= 6x + 2\frac{x^{-2}}{-2} - 12\frac{x^{-3}}{-3} + C \\ &= 6x - x^{-2} + 4x^{-3} + C \end{aligned}$$

8. Find a function $h(t)$ that satisfies $h''(t) = 12t - 2$, $h(0) = -1$ and $h(1) = 5$.

Solution: From $h''(t) = 12t - 2$, we get

$$h'(t) \quad = \quad 6t^2 - 2t + C_1 \tag{2}$$

and

$$h(t) \quad = \quad 2t^3 - t^2 + C_1 t + C_2 \tag{3}$$

Using (3) and the given fact that $h(0) = -1$, we get

$$\begin{aligned} -1 &= h(0) \\ &= C_2 \end{aligned}$$

so that $C_2 = -1$, and we have

$$h(t) \quad = \quad 2t^3 - t^2 + C_1 t - 1 \tag{4}$$

Using equation (4) and the fact that $h(1) = 5$, we get

$$\begin{aligned} 5 &= 2 - 1 + C_1 - 1 \\ &= C_1 \end{aligned}$$

so that $C_1 = 5$. We conclude that

$$h(t) \quad = \quad 2t^3 - t^2 + 5t - 1$$

9. Find the area under the curve $y = 4x - x^3$ between $x = 0$ and $x = 2$.

Solution: We have

$$\begin{aligned} \int_0^2 4x - x^3\, dx &= 2x^2 - \frac{x^4}{4}\Big]_0^2 \\ &= (8 - 4) - (0 - 0) \\ &= 4 \end{aligned}$$