

**Definition:** For  $a > 0$ ,  $a \neq 1$  and  $x > 0$ , we have

$$y = \log_a x \text{ if and only if } a^y = x.$$

More informally,  $\log_a x$  is *the power you raise  $a$  to to get  $x$* .

**Example:** Evaluate  $\log_3 \frac{1}{9}$ .

Solution: We note that  $\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$ . The power you raise 3 to to get  $\frac{1}{9}$  is  $-2$ , so we have

$$\log_3 \frac{1}{9} = -2.$$

**Definition:** We write  $\ln x$  for  $\log_e x$ , so that  $\ln x$  is *the power you raise  $e$  to to get  $x$* . This is called the *natural logarithm* function.

**Exercises I:** Evaluate

a)  $\log_2 16$

e)  $\log_5 5\sqrt{5}$

b)  $\log_4 \frac{1}{2}$

f)  $\log_{10} 100$

c)  $\log_5 125$

g)  $\log_{100} 10$

d)  $\log_7 1$

h)  $\ln \frac{1}{\sqrt{e}}$

**Cancellation Laws:** From the definition of the logarithm, we get the identities

$$\log_a a^x = x$$

$$\ln e^x = x$$

$$a^{\log_a x} = x \text{ (for } x > 0)$$

$$e^{\ln x} = x \text{ (for } x > 0)$$

We can use these identities to solve equations involving exponents and logarithms.

**Example:** Solve the equation  $3^{2x-2} = 27$  for  $x$ .

**Solution:** We take logarithms to the base 3, getting

$$\log_3 3^{2x-2} = \log_3 27.$$

By a cancellation law, the left side is equal to  $2x - 2$ . The right side is equal to 3, since  $3^3 = 27$ . Thus we have

$$2x - 2 = 3.$$

We solve this to get  $x = \frac{5}{2}$ .

**Example:** Solve the equation  $\ln(7 - 2x) = 3$  for  $x$ .

**Solution:** From the given equation, we get

$$e^{\ln(7-2x)} = e^3.$$

We apply a cancellation law to the left side to get

$$7 - 2x = e^3.$$

We solve this for  $x$ , getting  $x = \frac{7 - e^3}{2}$ . This is the exact form of the answer; if necessary, we can use a calculator to get the approximation  $x \approx -6.543$ .

**Exercises II:** Solve for  $x$ :

a)  $\log_2 x = -1$

d)  $4^{x-1} = \frac{1}{16}$

b)  $\log_5(1 - 4x) = \frac{1}{2}$

e)  $3^{1-2x} - \frac{\sqrt{3}}{3} = 0$

c)  $\ln(3x + 5) = -2$

f)  $e^{2x-5} = 8$

**Log Laws:** The following identities relate sums, differences, and multiples of logarithms to logarithms of products, quotients, and powers. As usual, we must have  $a > 0$  and  $a \neq 1$ , and any number whose logarithm we take must be positive.

$\log_a(xy) = \log_a x + \log_a y$	$\ln(xy) = \ln x + \ln y$
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\log_a(x^r) = r \log_a x$	$\ln(x^r) = r \ln x$

**Example:** Rewrite the expression  $\log_3 45 - \log_3 5$  as a single logarithm, and simplify it if possible.

**Solution:** By the second log law, we get

$$\begin{aligned}\log_3 45 - \log_3 5 &= \log_3 \left( \frac{45}{5} \right) \\ &= \log_3 9\end{aligned}$$

and since  $3^2 = 9$ , we know that this expression is equal to 2.

**Example:** Solve the equation  $\ln x + \ln(x + 2) - 1 = 0$  for  $x$ .

**Solution:** We use the first log law (for  $\ln$ ) to combine the two logarithms, getting

$$\begin{aligned}\ln(x(x + 2)) - 1 &= 0 \\ \ln(x^2 + 2x) &= 1.\end{aligned}$$

Next we exponentiate both sides (that is, raise  $e$  to both sides) to get

$$\begin{aligned}x^2 + 2x &= e \\ x^2 + 2x - e &= 0.\end{aligned}$$

We apply the quadratic formula to find two solutions

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 + 4e}}{2} \\ &= -1 \pm \sqrt{1 + e}\end{aligned}$$

In checking these answers, we observe that the root  $-1 - \sqrt{1 + e}$  is negative, and so  $\ln(-1 - \sqrt{1 + e})$  is undefined. We must therefore reject this solution. The other root,  $x = -1 + \sqrt{1 + e}$ , does satisfy the given equation.

**Exercises III:** Write as a single logarithm, and simplify if possible.

a)  $\log_2 10 + \log_2 12.8$

c)  $\frac{1}{2} \log_5 8 - \log_5 2$

b)  $\log_{10} 560 - \log_{10} 5.6$

d)  $\ln x - \frac{1}{2} \ln(x^2 + 1)$

Solve for  $x$ :

a)  $\log_3 x + \log_3 7 = 2$

b)  $\ln x - \ln(x^2 - 1) = 2$

**Change of base Formula:** For  $a > 0$ ,  $a \neq 1$ , and  $x > 0$ , we have the identity

$$\log_a x = \frac{\ln x}{\ln a}$$

We can use this to convert any logarithmic expression into an expression involving only the natural logarithm,  $\ln$ . Most calculators have a natural logarithm key, and the change of base formula gives us a handy way to approximate logarithms with arbitrary bases on a calculator.

**Example:** Use a calculator to approximate  $\log_3 7$ .

**Solution:** From the change of base formula, we get

$$\log_3 7 = \frac{\ln 7}{\ln 3}$$

The calculator says that this is approximately 1.77124.

**Example:** Solve the equation  $3^{12x-4} = 19$  for  $x$ , giving the answer in terms of natural logarithms.

**Solution:** We take logarithms to the base 3 to get

$$12x - 4 = \log_3 19$$

and solve for  $x$ , getting  $x = \frac{\log_3 19 + 4}{12} = \frac{1}{12} \log_3 19 + \frac{1}{3}$ . We use the change of base formula to put the answer in the required form,

$$x = \frac{\ln 19}{12 \ln 3} + \frac{1}{3}.$$

**Alternate solution:** We begin by taking natural logarithms, getting

$$\ln 3^{12x-4} = \ln 19$$

and then apply a log law to get

$$\begin{aligned} (12x - 4) \ln 3 &= \ln 19 \\ 12x - 4 &= \frac{\ln 19}{\ln 3} \\ 12x &= \frac{\ln 19}{\ln 3} + 4 \\ x &= \frac{\ln 19}{12 \ln 3} + \frac{1}{3}. \end{aligned}$$

**Exercises IV:** Solve each equation for  $x$ . Give the answer first in terms of natural logarithms, and then as a decimal approximation.

a)  $2^{1-4x} = 7$

c)  $3^x = 7 \cdot 9^{2x-5}$

b)  $10e^{1.5x} = 1500$

d)  $1.06^{\frac{x}{12}} = 2$