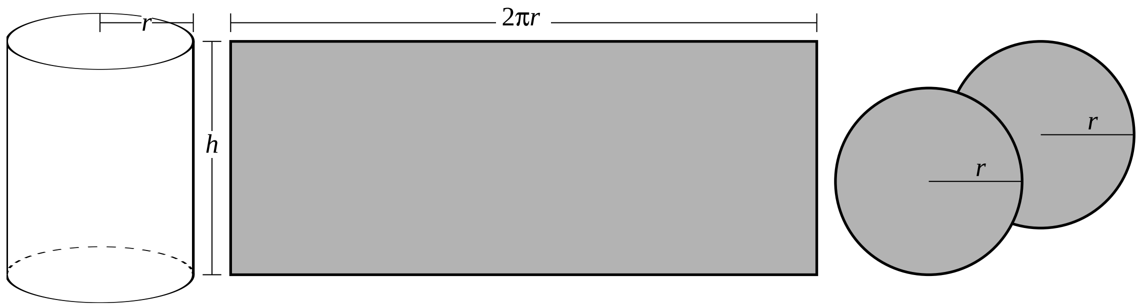
Optimization problems – Solutions

1. A cylindrical can is to have a volume of 400 cm^3. Find the dimensions (height and radius) of the can so as to minimize its total surface area. (The surface area comprises the top and bottom and the lateral surface.)

   Solution: Let r and h denote the radius and height of the can. Here is a sketch of the can and the material used to construct it.



From the given information, we have

$$400 = \pi r^2 h$$

so that $h = \dfrac{400}{\pi r^2}$. The lateral surface of the can is a rectangle with height h and width $2\pi r$. The top and bottom of the can are disks with radius r. The total surface area A of the can is given by

$$\begin{aligned} A &= 2(\pi r^2) + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \cdot \frac{400}{\pi r^2} \\ &= 2\pi r^2 + \frac{800}{r}. \end{aligned}$$

We need to minimize A, given that $r > 0$. We have

$$A' = 4\pi r - \frac{800}{r^2}$$

The critical numbers with $r > 0$ will satisfy the equation

$$4\pi r = \frac{800}{r^2}.$$

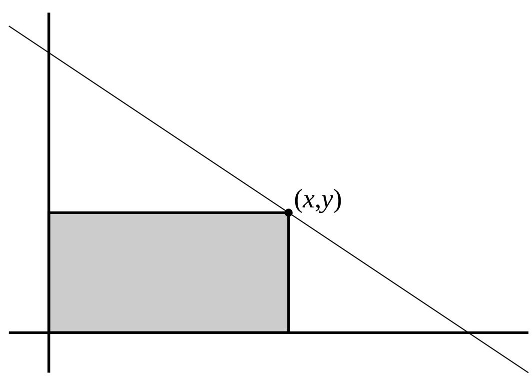
The only solution to this equation is $r = \sqrt[3]{\frac{200}{\pi}}$. We check that when r is near zero, $A'(r)$ is negative, and when r is large, $A'(r)$ is positive, so we know that A has a local and absolute minimum at $r = \sqrt[3]{\frac{200}{\pi}}$. The dimensions of the can with the minimum surface area are

$$\begin{aligned} r &= \sqrt[3]{\frac{200}{\pi}} \\ h &= \frac{400}{\pi\left(\frac{200}{\pi}\right)^{\frac{2}{3}}} \\ &= 2\sqrt[3]{\frac{200}{\pi}}. \end{aligned}$$

2. One corner of a rectangle is at the origin in the xy-plane, and the opposite corner lies in the first quadrant, along the line $y = 7 - \frac{2}{3}x$. The sides of the rectangle are parallel to the coordinate axes. Find the largest possible area of such a rectangle.

Solution: Let (x, y) be the coordinates of the rectangle's upper right corner. Then the area of the rectangle is given by

$$A = xy.$$



We also know that $y = 7 - \frac{2}{3}x$, so we can take

$$\begin{aligned} A &= x\left(7 - \frac{2}{3}x\right) \\ &= 7x - \frac{2}{3}x^2. \end{aligned}$$

We get

$$A' = 7 - \frac{4}{3}x$$

so that A has only one critical number at

$$x = \frac{21}{4}.$$

It is easy to see that A' is positive for values of x less than $\frac{21}{4}$ and negative for values of x greater than $\frac{21}{4}$, so that we have a local maximum at $x = \frac{21}{4}$. The area of the largest rectangle is

$$\frac{21}{4}\left(7 - \frac{2}{3}\frac{21}{4}\right) = \frac{147}{8}.$$

3. Find the positive number x for which $5x + \frac{1}{x^2}$ is as small as possible.

   Solution: Let $f(x) = 5x + \frac{1}{x^2}$. We need to minimize $f(x)$ with $x > 0$. We get

$$f'(x) = 5 - \frac{2}{x^3}.$$

   The only cricital number is the solution to

$$\begin{aligned} 5 &= \frac{2}{x^3} \\ x^3 &= \frac{2}{5}. \end{aligned}$$

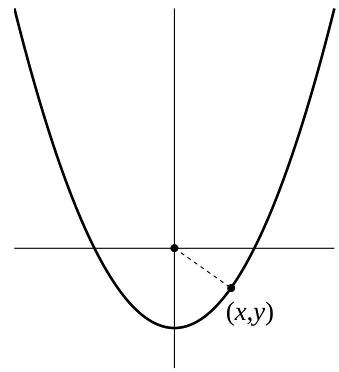
   It is easy to check that $f'(x)$ is negative for $x < \sqrt[3]{\frac{2}{5}}$ and $f'(x)$ is positive for $x > \sqrt[3]{\frac{2}{5}}$, so we have a local (and absolute) minimum. The function f is minimized when

$$x = \sqrt[3]{\frac{2}{5}}.$$

4. Find the points on the curve $y = x^2 - 1$ that are closest to the origin.

Solution: Let (x, y) be a point on the curve. We want to minimize the distance from (x, y) to the origin. We may as well minimize the square of the distance to the origin, that is

$$D = x^2 + y^2.$$



We know that $y = x^2 - 1$, so we can write

$$D = x^2 + (x^2 - 1)^2.$$

Taking a derivative, we get

$$\begin{aligned} D' &= 2x + 4x(x^2 - 1) \\ &= 2x(1 + 2(x^2 - 1)) \\ &= 2x(2x^2 - 1). \end{aligned}$$

The critical numbers are $x = 0$ and $x = \pm\sqrt{\frac{1}{2}}$. Here is a sign chart for D':

	$x < -\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}} < x < 0$	$0 < x < \sqrt{\frac{1}{2}}$	$x > \sqrt{\frac{1}{2}}$
$2x$:	$-$	$-$	$+$	$+$
$2x^2 - 1$:	$+$	$-$	$-$	$+$
$D'(x)$:	$-$	$+$	$-$	$+$

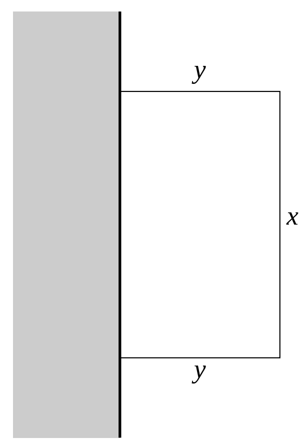
There is a local maximum at $x = 0$ and there are local (and absolute) minima at $x = \pm\sqrt{\frac{1}{2}}$. The points on the curve that are closest to the

origin are

$$\left(\pm\sqrt{\frac{1}{2}}, -\frac{1}{2}\right)$$

5. A rectangular garden with an area of 1000 square meters is to be laid out beside a straight river. The bank of the river provides one side of the garden; along the other three sides we need to put up fencing. Find the minimum amount of fencing needed to enclose the garden.

Solution: Let x denote the length of the side of the garden parallel to the river and let y denote the length of the adjacent side. Let P denote the amount of fencing we will need. Then we have



$$P = x + 2y,$$

and our goal is to minimize P. We also know that the area of the garden, xy, is equal to 1000, so we have

$$\begin{aligned} xy &= 1000 \\ y &= \frac{1000}{x}. \end{aligned}$$

Substituting for y in the expression for P, we get

$$P = x + \frac{2000}{x}.$$

We find that

$$P' = 1 - \frac{2000}{x^2}.$$

so the critical number of P (with $x > 0$) satisfies

$$1 = \frac{2000}{x^2}.$$

Thus $x = \sqrt{2000} = 20\sqrt{5}$ is a critical number for P. Here is a sign chart for P':

$$\begin{array}{ccc} - & & + \\ \hline & 20\sqrt{5} & \end{array}$$

We have an absolute minimum at $x = 20\sqrt{5}$. Thus for the minimum amount of fencing, we take

$$x = 20\sqrt{5} \quad \text{and} \quad y = \frac{1000}{x} = \frac{50}{\sqrt{5}} = 10\sqrt{5}.$$

The minimum amount of fencing is given by

$$x + 2y \quad = \quad 40\sqrt{5} \text{ meters.}$$