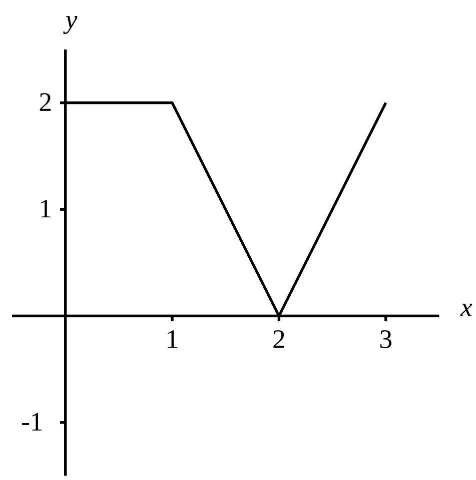
Math 101(02) Solution
Practice Exam I 23 September 2003

1. Let f be the function with domain $[0, 3]$ given by

$$f(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1 \\ |4 - 2x| & \text{if } 1 < x \leq 3 \end{cases}$$

(a) Sketch a graph of f on one of the grids provided. (The other grid is for practice; be sure to indicate which one you want to count as your answer.)

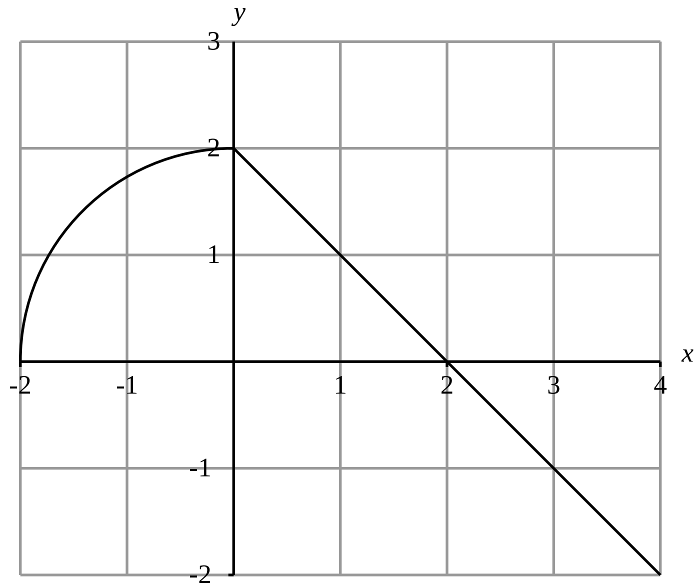
Solution:



(b) Find $f \circ f\left(\frac{1}{2}\right)$.

Solution: We have $f\left(f\left(\frac{1}{2}\right)\right) = f(2) = 0$.

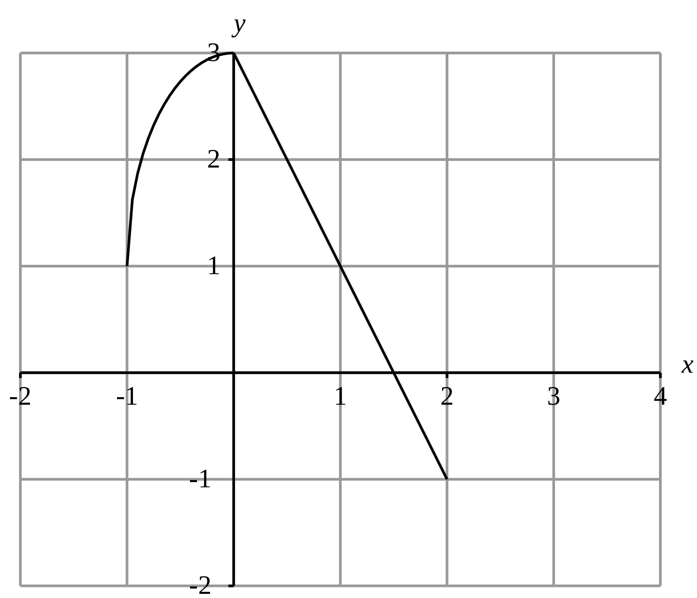
2. Here is the graph of a function f, defined on the interval $[-2, 4]$.



On the blank grid below, draw a graph of the function g given by

$$g(x) \quad = \quad 1 + f(2x).$$

Solution:



3. An air taxi service based at Barnes Airport can get a passenger from the gate at Barnes to the gate at an airport 105 miles away in one hour and thirty-five minutes. Using the same airplane, the service can get a passenger from the gate at Barnes to the gate at an airport 216 miles away in two hours and forty-nine minutes.

   (a) Assuming a linear model is appropriate, find T, the gate-to-gate time for an air taxi trip, as a function of x, the distance from Barnes to the destination airport.
   Solution: We know that $T = 95$ (in minutes) when $x = 105$ (in miles), and $T = 169$ when $x = 216$. The slope of the linear function we want is thus

$$\begin{aligned} \frac{\text{rise}}{\text{run}} &= \frac{169 - 95}{216 - 105} \\ &= \frac{74}{111} \\ &= \frac{2}{3}. \end{aligned}$$

   We can find the equation of the line using the point-slope form. We have

$$T - 95 = \frac{2}{3}(x - 105).$$

   This simplifies to

$$T = \frac{2}{3}x + 25.$$

   The function we want is

$$T(x) = \frac{2}{3}x + 25$$

   where x is in miles and T is in minutes.

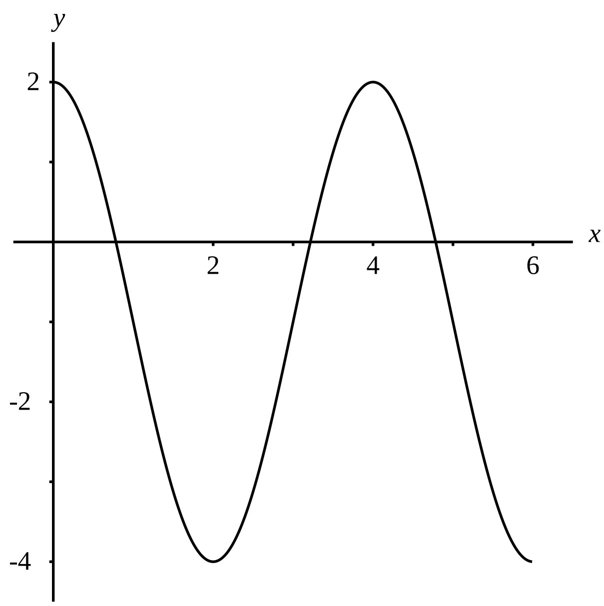
   (b) What is the value of the slope of the line in part 3a, and what is its significance?
   Solution: The slope is $\frac{2}{3}$. The units of the slope are minutes per mile. The significance is that for each additional mile added to the trip, the travel time is increased by $\frac{2}{3}$ minutes.

   This suggests that the airplane's cruising speed is $\frac{3}{2}$ miles per minute, or 90 miles per hour.

(c) What is the value of the T-intercept in part 3a, and what is its significance?

Solution: The T-intercept is 25, measured in minutes. This is the amount of time it would take for the air taxi passenger to complete a trip of zero miles. How could a trip of zero miles take 25 minutes? One possibility is that the 25 minutes is "overhead" time: time to get the passenger seated, time for taxi and take-off, maneuvering for landing, taxiing in at the destination airport, and walking the passenger to the arrival gate.

4. Below is the graph of the function $A + B\cos(Cx)$. Determine the values of A, B, and C.



Solution: The mean value is -1, the amplitude is 3 and the period is 4. The function graphed is

$$y = -1 + 3\cos\left(\frac{\pi x}{2}\right).$$

5. Let f be the function given by $f(x) = \sqrt{x+2}$.

(a) What is the largest possible domain for f?
Solution: The expression $\sqrt{x+2}$ is defined whenever $x+2 \geq 0$, that is, when $x \geq -2$. So the largest possible domain for f is $[-2, \infty)$.

(b) Find a formula for $f^{-1}(x)$.
Solution: We write

$$x = \sqrt{y+2}$$

and solve for y. We get

$$\begin{aligned} x^2 &= y+2 \\ y &= x^2 - 2. \end{aligned}$$

Thus $f^{-1}(x) = x^2 - 2$.

(c) What is the domain of f^{-1}?
Solution: We know that in general the domain of the inverse of any function is the same as the range of the original function. The range of f in this problem is $[0, \infty)$, so that is the domain of f^{-1}.

6. (a) Solve the equation $5^{2-x} - 3 = 0$ for x.
Write your answer in terms of natural logarithms.
Solution: We have $5^{2-x} = 3$. Taking the natural log of both sides we get

$$\begin{aligned}(2-x)\ln 5 &= \ln 3\\ 2\ln 5 - \ln 3 &= x\ln 5\\ x &= \frac{2\ln 5 - \ln 3}{\ln 5}\\ &= 2 - \frac{\ln 3}{\ln 5}.\end{aligned}$$

(b) Solve the equation $\ln x + \ln(x-2) = 3$ for x.
Write your answer in exact form.
Solution: We have

$$\ln(x^2 - 2x) = 3.$$

We exponentiate both sides to get

$$\begin{aligned}x^2 - 2x &= e^3\\ x^2 - 2x - e^3 &= 0.\end{aligned}$$

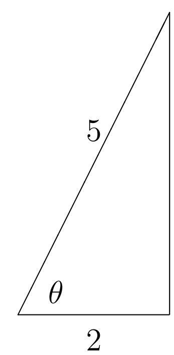
From the quadratic formula, we get

$$\begin{aligned}x &= \frac{2 \pm \sqrt{4+4e^3}}{2}\\ &= 1 \pm \sqrt{1+e^3}.\end{aligned}$$

Since $1 - \sqrt{1+e^3}$ is negative, its natural log is undefined, so we must reject that solution. The other solution is valid, though, so we have $x = 1 + \sqrt{1+e^3}$.

7. (a) Find the exact value of $\sin\left(\cos^{-1}\left(\frac{2}{5}\right)\right)$.

Solution: Let θ denote $\cos^{-1}\left(\frac{2}{5}\right)$. Then θ is the indicated angle in the right triangle at right.
The third side of this triangle has length $\sqrt{25-4} = \sqrt{21}$, so the sine of θ is $\frac{\sqrt{21}}{5}$.



(b) Find the exact value of $\cos^{-1}\left(\cos\left(\frac{13\pi}{10}\right)\right)$.
Give your answer in radians.
Solution: In the unit circle picture at right, the darker of the two rays makes an angle of $\frac{13\pi}{10}$ with the positive x-axis, so it intersects the circle in the point $\left(\cos\left(\frac{13\pi}{10}\right), \sin\left(\frac{13\pi}{10}\right)\right)$.
The angle between 0 and π that has the same cosine is indicated by the lighter ray. By symmetry, this angle must be $\frac{3\pi}{10}$ radians above the angle π, so its measure is $\frac{7\pi}{10}$.