1. Let $f$ be the function on the interval $[-2, 3]$ given by

$$f(x) = \begin{cases} 
  x + 3 & \text{if } -2 \leq x < 0 \\
  2 - \frac{x}{2} & \text{if } 0 \leq x < 2 \\
  1 - (x - 2)^2 & \text{if } 2 \leq x < 3 
\end{cases}$$

(a) Find $\lim_{x \to 0} f(x)$.

(b) Find $\lim_{x \to 2} f(x)$.

(c) Find all numbers $x$ in $[-2, 3]$ at which $f$ is not differentiable.

Solution: A graph of $f$ will be helpful. Here it is:

(a) From the graph, it’s clear that $\lim_{x \to 0^-} f(x) = 3$ and $\lim_{x \to 0^+} f(x) = 2$, so that $\lim_{x \to 0} f(x)$ does not exist.
(b) The left- and right-hand limits do agree at \( x = 2 \), so we have \( \lim_{x \to 2} f(x) = 1 \).

(c) The function \( f \) fails to be differentiable at \( x = 0 \), because it is discontinuous there. It also fails to be differentiable at \( x = 2 \), because the slope coming into \( x = 2 \) from the left is \( -\frac{1}{2} \) and the slope coming in from the right is 0. It’s subtle, but there is a corner in the graph at \( x = 2 \).
2. Find the following limits algebraically. Show your work.

(a) \( \lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - 4} \)

Solution: We factor the numerator and denominator to get

\[
\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \to 2} \frac{(x + 4)(x - 2)}{(x - 2)(x + 2)} \\
= \lim_{x \to 2} \frac{x + 4}{x + 2} \\
= \frac{3}{2}.
\]

(b) \( \lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} \)

Solution: We factor the numerator and denominator to get

\[
\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \to -2^-} \frac{(x + 4)(x - 2)}{(x - 2)(x + 2)} \\
= \lim_{x \to -2^-} \frac{x + 4}{x + 2}.
\]

As \( x \) approaches \(-2\) from the left, the numerator approaches 2 and the denominator approaches 0 from the left, so the quotient goes to \(-\infty\).
(c) \( \lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 + 1}) \)

Solution:

\[
\lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 + 1}) = \lim_{x \to \infty} \frac{(x^2 + x) - (x^2 + 1)}{\sqrt{x^2 + x} + \sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{x - 1}{\sqrt{x^2 + x} + \sqrt{x^2 + 1}}.
\]

For positive values of \( x \), we know that \( x = \sqrt{x^2} \), so we divide the top of this fraction by \( x \) and the bottom by \( \sqrt{x^2} \) to get

\[
\lim_{x \to \infty} \frac{x - 1}{\sqrt{x^2 + x} + \sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1 - \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}}} = \frac{1}{\sqrt{1 + \sqrt{1}}} = \frac{1}{2}.
\]

(d) \( \lim_{x \to -\infty} \frac{\sqrt{2x^4 + 8}}{(x - 3)(x - 5)} \)

Solution: For \( x \to -\infty \) we know that \( x = -\sqrt{x^2} \), but in this problem we want to know about \( \sqrt{x^4} \). We know it must be \( \pm x^2 \), but which one? Since \( x^2 \) is always non-negative, it must be that \( x^2 \) is the positive square root of \( x^4 \). That is, \( \sqrt{x^4} = x^2 \), even though \( x \) is negative.

We divide the top of the given fraction by \( \sqrt{x^4} \) and the bottom by \( x^2 \) to get

\[
\lim_{x \to -\infty} \frac{\sqrt{2x^4 + 8}}{x^2 - 8x + 15} = \lim_{x \to -\infty} \frac{\sqrt{2 + \frac{8}{x^4}}}{1 - \frac{8}{x} + \frac{15}{x^2}} = \sqrt{2}.
\]
3. Use the definition of the derivative to find \( f'(x) \) if \( f(x) = \frac{1}{\sqrt{x}} \).

Solution: We have

\[
f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}
= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x} \sqrt{x+h})}.
\]

We multiply top and bottom by \( \sqrt{x} + \sqrt{x+h} \) to get

\[
f'(x) = \lim_{h \to 0} \frac{x - (x+h)}{h \sqrt{x} \sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}
= \lim_{h \to 0} \frac{-h}{h \sqrt{x} \sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}.
\]

Since we are looking at a limit, we may cancel the \( h \) from the top and bottom to get

\[
f'(x) = \lim_{h \to 0} \frac{-1}{\sqrt{x} \sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}
= \frac{1}{\sqrt{x} \sqrt{x} (\sqrt{x} + \sqrt{x})}
= \frac{1}{2x\sqrt{x}}.
\]
4. Use any appropriate differentiation rules to compute $f'(x)$.

(a) $f(x) = \frac{3^x}{x}$

Solution: We apply the quotient rule to get

$$f'(x) = \frac{x3^x \ln 3 - 3^x}{x^2} = \frac{(x \ln 3 - 1)3^x}{x^2}.$$ 

(b) $f(x) = \frac{\sin(2x)}{\sqrt{x}}$

Solution: The quotient and chain rules apply. We get

$$f'(x) = \frac{2\sqrt{x} \cos(2x) - \frac{1}{2}x^{-\frac{1}{2}} \sin(2x)}{x} = \frac{2 \cos(2x)}{\sqrt{x}} - \frac{\sin(2x)}{x^\frac{3}{2}}.$$ 

(c) $f(x) = x^3(2x^2 + 7)^{10}$

Solution: We have

$$f'(x) = 3x^2 \cdot (2x^2 + 7)^{10} + x^3 \cdot 10(2x^2 + 7)^9 \cdot (4x) = 3x^2(2x^2 + 7)^{10} + 40x^4(2x^2 + 7)^9.$$
5. Suppose we invest $1000 at a fixed, continuously-compounded interest rate $r$ for ten years. The value of the investment at the end of the ten years depends on the interest rate $r$. Let $P(r)$ denote the value of the investment (in dollars) at the end of ten years when the interest rate is $r$ percent.

Suppose that $P(3) = 1350$ and $P'(3) = 135$.

(a) What are the units of $P'(r)$? What is the meaning of the statement $P'(3) = 135$?
Solution: The units of $P'(r)$ are dollars per percentage point. The statement $P'(3) = 135$ tells us that for each increase of one percentage point in $r$ (for $r$ near 3), the final value of the investment increases by about 135 dollars.

(b) What is a good estimate for $P(3.3)$?
Solution: As $r$ increases from 3.0 to 3.3, $P(r)$ should increase by about $0.3 \times P'(3)$, which is 40.5. We expect that $P(3.3)$ will be about 1390.5 dollars.
6. The line in the picture below crosses the $x$-axis at the point $(-4,0)$ and is tangent to the parabola $y = 5x - x^2$. Find an equation for the line. (The picture is not to scale.)

Solution: Let $a$ denote the $x$-coordinate of the point of tangency. Then the line passes through the points $(a, 5a - a^2)$ and $(-4, 0)$. So the slope $m$ of the line is $\frac{5a - a^2}{a + 4}$.

Since $y' = 5 - 2x$ along the parabola and the line is tangent to the parabola at the point where $x = a$, we know that $m = 5 - 2a$. Our two expressions for the slope of the line must be equal. That is,

$$\frac{5a - a^2}{a + 4} = 5 - 2a.$$

We can solve this equation for $a$. We get

$$5a - a^2 = (a + 4)(5 - 2a)$$

$$5a - a^2 = 5a - 2a^2 - 8a + 20$$

$$a^2 + 8a - 20 = 0$$

$$(a + 10)(a - 2) = 0.$$ 

The solutions are $a = -10$ and $a = 2$. The point of tangency in the picture clearly has a positive $x$-coordinate, so its $x$-coordinate must be 2. The $y$-coordinate is $5(2) - 2^2 = 6$.

The line passes through $(-4, 0)$ and $(2, 6)$, so its slope is $6/6 = 1$. Using the point-slope form of a line, we get

$$y - 6 = x - 2$$

or $y = x + 4$. 

7. Suppose that \( f \) is a differentiable function, and \( G(x) = f(x^3 - 7) \). Given that \( G(2) = 10 \) and \( G'(2) = -3 \), find \( f(1) \) and \( f'(1) \).

Solution: First of all, the information that \( G(2) = 10 \) says that

\[
10 = f(2^3 - 7) \\
= f(8 - 7) \\
= f(1).
\]

So \( f(1) = 10 \).

To find \( f'(1) \) from \( G'(2) \), we recall that the chain rule says

\[
G'(x) = f'(x^3 - 7) \cdot (3x^2).
\]

Evaluating this at \( x = 2 \) we get

\[
G'(2) = f'(2^3 - 7) \cdot 3(2^2) \\
= f'(1) \cdot 12
\]

so that \( f'(1) = \frac{G'(2)}{12} \). Since we know that \( G'(2) = -3 \), we can compute \( f'(1) \). We get

\[
f'(1) = -\frac{1}{4}.
\]