Math 101(02)
Practice Exam II

Solution
20 October 2003

1. Let f be the function on the interval $[-2,3]$ given by

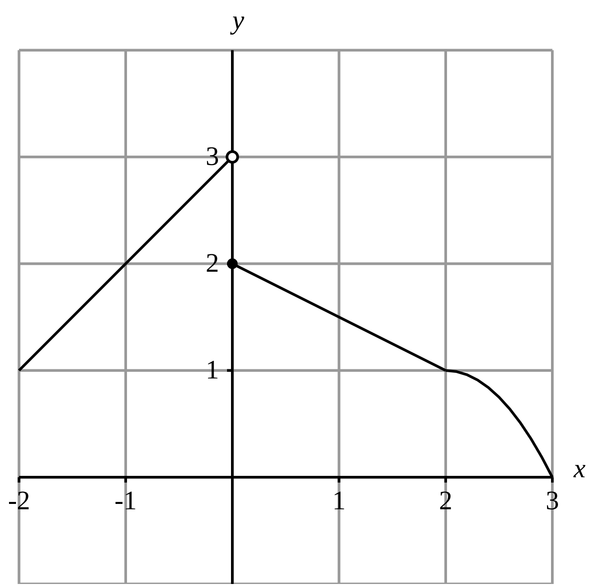
$$f(x) = \begin{cases} x+3 & \text{if } -2 \le x < 0 \\ 2 - \dfrac{x}{2} & \text{if } 0 \le x < 2 \\ 1-(x-2)^2 & \text{if } 2 \le x < 3 \end{cases}$$

(a) Find $\lim_{x\to 0} f(x)$.

(b) Find $\lim_{x\to 2} f(x)$.

(c) Find all numbers x in $[-2,3]$ at which f is not differentiable.

Solution: A graph of f will be helpful. Here it is:



(a) From the graph, it's clear that $\lim_{x\to 0^-} f(x) = 3$ and $\lim_{x\to 0^+} f(x) = 2$, so that $\lim_{x\to 0} f(x)$ does not exist.

(b) The left- and right-hand limits do agree at $x = 2$, so we have $\lim_{x \to 2} f(x) = 1$.

(c) The function f fails to be differentiable at $x = 0$, because it is discontinuous there. It also fails to be differentiable at $x = 2$, because the slope coming into $x = 2$ from the left is $-\frac{1}{2}$ and the slope coming in from the right is 0. It's subtle, but there is a corner in the graph at $x = 2$.

2. Find the following limits algebraically. Show your work.

(a) $\lim_{x\to 2} \dfrac{x^2+2x-8}{x^2-4}$

Solution: We factor the numerator and denominator to get

$$\begin{aligned}\lim_{x\to 2} \frac{x^2+2x-8}{x^2-4} &= \lim_{x\to 2} \frac{(x+4)(x-2)}{(x-2)(x+2)} \\ &= \lim_{x\to 2} \frac{x+4}{x+2} \\ &= \frac{3}{2}.\end{aligned}$$

(b) $\lim_{x\to -2^-} \dfrac{x^2+2x-8}{x^2-4}$

Solution: We factor the numerator and denominator to get

$$\begin{aligned}\lim_{x\to -2^-} \frac{x^2+2x-8}{x^2-4} &= \lim_{x\to -2^-} \frac{(x+4)(x-2)}{(x-2)(x+2)} \\ &= \lim_{x\to -2^-} \frac{x+4}{x+2}.\end{aligned}$$

As x approaches -2 from the left, the numerator approaches 2 and the denominator approaches 0 from the left, so the quotient goes to $-\infty$.

(c) $\lim_{x\to\infty}(\sqrt{x^2+x}-\sqrt{x^2+1})$

Solution:

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+x}-\sqrt{x^2+1}) &= \lim_{x\to\infty}\frac{(x^2+x)-(x^2+1)}{\sqrt{x^2+x}+\sqrt{x^2+1}}\\ &= \lim_{x\to\infty}\frac{x-1}{\sqrt{x^2+x}+\sqrt{x^2+1}}.\end{aligned}$$

For positive values of x, we know that $x=\sqrt{x^2}$, so we divide the top of this fraction by x and the bottom by $\sqrt{x^2}$ to get

$$\begin{aligned}\lim_{x\to\infty}\frac{x-1}{\sqrt{x^2+x}+\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{1-1/x}{\sqrt{1+1/x}+\sqrt{1+1/x^2}}\\ &= \frac{1}{\sqrt{1}+\sqrt{1}}\\ &= \frac{1}{2}.\end{aligned}$$

(d) $\lim_{x\to-\infty}\frac{\sqrt{2x^4+8}}{(x-3)(x-5)}$

Solution: For $x\to-\infty$ we know that $x=-\sqrt{x^2}$, but in this problem we want to know about $\sqrt{x^4}$. We know it must be $\pm x^2$, but which one? Since x^2 is always non-negative, it must be that x^2 is the positive square root of x^4. That is, $\sqrt{x^4}=x^2$, even though x is negative.

We divide the top of the given fraction by $\sqrt{x^4}$ and the bottom by x^2 to get

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{2x^4+8}}{x^2-8x+15} &= \lim_{x\to-\infty}\frac{\sqrt{2+8/x^4}}{1-8/x+15/x^2}\\ &= \sqrt{2}.\end{aligned}$$

3. Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt{x}}$.

Solution: We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}. \end{aligned}$$

We multiply top and bottom by $\sqrt{x} + \sqrt{x+h}$ to get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\ &= \lim_{h\to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}. \end{aligned}$$

Since we are looking at a limit, we may cancel the h from the top and bottom to get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\ &= -\frac{1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} \\ &= -\frac{1}{2x\sqrt{x}}. \end{aligned}$$

4. Use any appropriate differentiation rules to compute $f'(x)$.

(a) $f(x) = \dfrac{3^x}{x}$

Solution: We apply the quotient rule to get

$$\begin{aligned} f'(x) &= \frac{x3^x \ln 3 - 3^x}{x^2} \\ &= \frac{(x \ln 3 - 1)3^x}{x^2}. \end{aligned}$$

(b) $f(x) = \dfrac{\sin(2x)}{\sqrt{x}}$

Solution: The quotient and chain rules apply. We get

$$\begin{aligned} f'(x) &= \frac{2\sqrt{x}\cos(2x) - \frac{1}{2}x^{-\frac{1}{2}}\sin(2x)}{x} \\ &= \frac{2\cos(2x)}{\sqrt{x}} - \frac{\sin(2x)}{x^{\frac{3}{2}}}. \end{aligned}$$

(c) $f(x) = x^3(2x^2+7)^{10}$

Solution: We have

$$\begin{aligned} f'(x) &= 3x^2 \cdot (2x^2+7)^{10} + x^3 \cdot 10(2x^2+7)^9 \cdot (4x) \\ &= 3x^2(2x^2+7)^{10} + 40x^4(2x^2+7)^9 \end{aligned}$$

5. Suppose we invest \$1000 at a fixed, continuously-compounded interest rate r for ten years. The value of the investment at the end of the ten years depends on the interest rate r. Let $P(r)$ denote the value of the investment (in dollars) at the end of ten years when the interest rate is r percent.

   Suppose that $P(3) = 1350$ and $P'(3) = 135$.

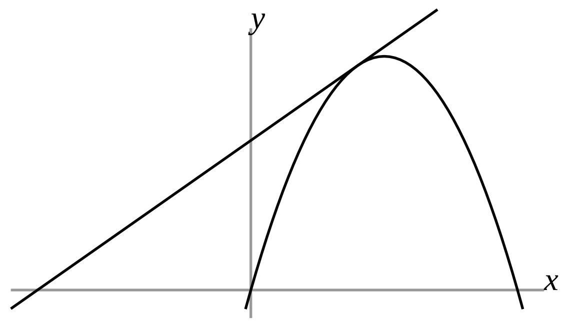
   (a) What are the units of $P'(r)$? What is the meaning of the statement $P'(3) = 135$?

   Solution: The units of $P'(r)$ are dollars per percentage point. The statement $P'(3) = 135$ tells us that for each increase of one percentage point in r (for r near 3), the final value of the investment increases by about 135 dollars.

   (b) What is a good estimate for $P(3.3)$?

   Solution: As r increases from 3.0 to 3.3, $P(r)$ should increase by about $0.3 \times P'(3)$, which is 40.5. We expect that $P(3.3)$ will be about 1390.5 dollars.

6. The line in the picture below crosses the x-axis at the point $(-4, 0)$ and is tangent to the parabola $y = 5x - x^2$. Find an equation for the line. (The picture is not to scale.)



Solution: Let a denote the x-coordinate of the point of tangency. Then the line passes through the points $(a, 5a - a^2)$ and $(-4, 0)$. So the slope m of the line is $\dfrac{5a - a^2}{a + 4}$.

Since $y' = 5 - 2x$ along the parabola and the line is tangent to the parabola at the point where $x = a$, we know that $m = 5 - 2a$. Our two expressions for the slope of the line must be equal. That is,

$$\frac{5a - a^2}{a + 4} = 5 - 2a.$$

We can solve this equation for a. We get

$$\begin{aligned} 5a - a^2 &= (a + 4)(5 - 2a) \\ 5a - a^2 &= 5a - 2a^2 - 8a + 20 \\ a^2 + 8a - 20 &= 0 \\ (a + 10)(a - 2) &= 0. \end{aligned}$$

The solutions are $a = -10$ and $a = 2$. The point of tangency in the picture clearly has a positive x-coordinate, so its x-coordinate must be 2. The y-coordinate is $5(2) - 2^2 = 6$.

The line passes through $(-4, 0)$ and $(2, 6)$, so its slope is $6/6 = 1$. Using the point-slope form of a line, we get

$$y - 6 = x - 2$$

or $y = x + 4$.

7. Suppose that f is a differentiable function, and $G(x) = f(x^3 - 7)$. Given that $G(2) = 10$ and $G'(2) = -3$, find $f(1)$ and $f'(1)$.

Solution: First of all, the information that $G(2) = 10$ says that

$$\begin{aligned} 10 &= f(2^3 - 7) \\ &= f(8 - 7) \\ &= f(1). \end{aligned}$$

So $f(1) = 10$.

To find $f'(1)$ from $G'(2)$, we recall that the chain rule says

$$G'(x) = f'(x^3 - 7) \cdot (3x^2).$$

Evaluating this at $x = 2$ we get

$$\begin{aligned} G'(2) &= f'(2^3 - 7) \cdot 3(2^2) \\ &= f'(1) \cdot 12 \end{aligned}$$

so that $f'(1) = \dfrac{G'(2)}{12}$. Since we know that $G'(2) = -3$, we can compute $f'(1)$. We get

$$f'(1) = -\frac{1}{4}.$$