1. Find y'.

   (a) $y = \sqrt{\sin^{-1}(2x)}$
   Solution: We have

$$y' = \frac{1}{2}(\sin^{-1}(2x))^{-\frac{1}{2}} \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$
$$= \frac{1}{\sqrt{\sin^{-1}(2x)}\sqrt{1-4x^2}}.$$

   (b) $y = \log_3(x^2+5)$
   Solution: We have

$$y' = \frac{1}{(\ln 3)(x^2+5)}(2x)$$
$$= \frac{2x}{(\ln 3)(x^2+5)}$$

   (c) $y = (\cos x)^x$ (Write your answer in terms of x.)
   Solution: We write

$$\ln y = x \ln(\cos x)$$

   and differentiate to get

$$\frac{y'}{y} = x\left(\frac{-\sin x}{\cos x}\right) + \ln(\cos x)$$
$$= -x \tan x + \ln(\cos x)$$

   So that

$$y' = y(-x\tan x + \ln(\cos x))$$
$$= (\cos x)^x(\ln(\cos x) - x\tan x)$$

2. Find the equations for the tangent lines to the following curves at the given points.

   (a) $xy^3 + x^3y^2 - 3x = 12$ at the point $(2, -3)$
   Solution: We differentiate implicitly to get

$$\begin{aligned} 3xy^2\frac{dy}{dx} + y^3 + 2yx^3\frac{dy}{dx} + 3x^2y^2 - 3 &= 0 \\ (3xy^2 + 2yx^3)\frac{dy}{dx} &= 3 - y^3 - 3x^2y^2 \end{aligned}$$

   so that $\dfrac{dy}{dx} = \dfrac{3 - y^3 - 3x^2y^2}{3xy^2 + 2yx^3}$. At the point $(2, -3)$, this gives $\dfrac{dy}{dx} = -13$. An equation for the tangent line is

$$y + 3 = -13(x - 2)$$

   (b) $y = \dfrac{(x^3 - 7)^5\sqrt{3x^2 + 13}}{x^2 + 1}$ at the point $(2, 1)$
   Solution: We use logarithmic differentiation to find $y'(2)$. We have

$$\ln y = 5\ln(x^3 - 7) + \frac{1}{2}\ln(3x^2 + 13) - \ln(x^2 + 1)$$

   so that

$$\frac{y'}{y} = \frac{5 \cdot 3x^2}{x^3 - 7} + \frac{6x}{2(3x^2 + 13)} - \frac{2x}{x^2 + 1}.$$

   We substitute in $x = 2$ and $y = 1$ to get

$$\begin{aligned} \frac{y'(1)}{1} &= \frac{60}{1} + \frac{12}{50} - \frac{4}{5} \\ &= \frac{2792}{50} \\ &= \frac{1486}{25} \end{aligned}$$

   so that $y'(2) = \dfrac{1486}{25}$. An equation for the tangent line is

$$y - 1 = \frac{1486}{25}(x - 2)$$

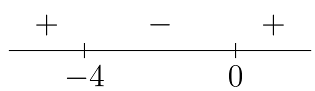
3. A particle moves along the x-axis with position function given by $x(t) = t^3 + 6t^2 - 9$.

Find the time intervals in which the particle is both moving to the left and slowing down.

Solution: We need to determine the times at which the particle has negative velocity and positive acceleration. For the velocity, we have

$$\begin{aligned} x'(t) &= 3t^2 + 12t \\ &= 3t(t+4) \end{aligned}$$

The velocity changes sign (and thus the particle changes direction) when $t = 0$ and when $t = -4$. Here is a sign chart for $x'(t)$.



The particle is moving to the left for $-4 < t < 0$.

The particle's acceleration is given by $x''(t)$. We have

$$\begin{aligned} x''(t) &= 6t + 12 \\ &= 6(t+2) \end{aligned}$$

This is clearly negative when $t < -2$ and positive when $t > -2$.

The time interval on which the particle has both negative velocity and positive acceleration is $-2 < t < 0$.

4. Find the linearization $L(x)$ of the function $f(x) = \sqrt{25 - x}$ at $a = 9$, and use it to approximate $\sqrt{25 - 10}$.

Solution: We have $f'(x) = -\frac{1}{2}(25 - x)^{-\frac{1}{2}}$, so that $f'(9) = -\frac{1}{2}(16)^{-\frac{1}{2}} = -\frac{1}{8}$. We also know that $f(9) = \sqrt{25 - 9} = 4$. The linearization we want is given by

$$L(x) = 4 - \frac{1}{8}(x - 9)$$

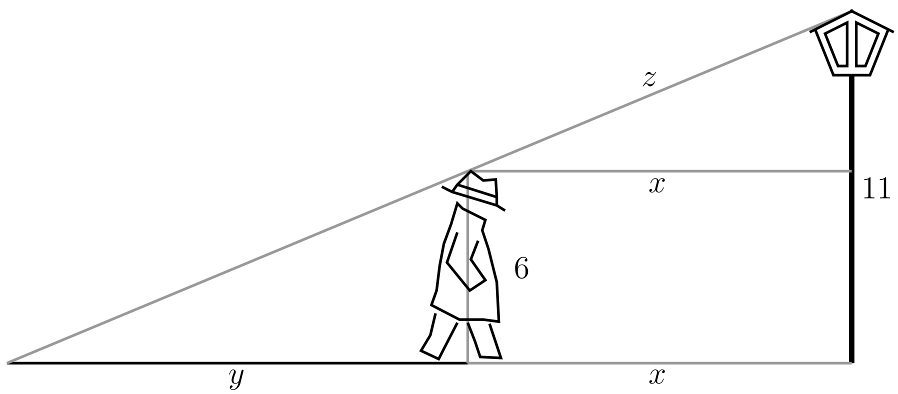
We approximate $\sqrt{25 - 10}$ (that is, $\sqrt{15}$), as

$$\begin{aligned} L(10) &= 4 - \frac{1}{8}(10 - 9) \\ &= 3.875 \end{aligned}$$

5. On a dark, still night, Miles Archer, who is six feet tall, walks directly toward his favorite lamppost, which is eleven feet tall. Miles walks at a steady (if deliberate) pace of four feet per second. His shadow silently trails him, getting shorter and shorter as he approaches the lamppost.

   (a) At what rate is Miles's shadow shrinking when he is twelve feet from the lamppost?

   Solution: Here is a diagram with the dimensions we'll need.



Let x denote the distance from Miles to the lamppost, and let y denote the length of Miles's shadow. By similar triangles, we have

$$\frac{6}{y} = \frac{11}{x+y}$$

We cross-multiply to get $11y = 6(x+y)$, or $5y = 6x$. We treat both x and y as functions of t and differentiate to get $5\frac{dy}{dt} = 6\frac{dx}{dt}$. We know that $\frac{dx}{dt} = -4$, so we easily find that

$$\begin{aligned} \frac{dy}{dt} &= \frac{6}{5} \times (-4) \\ &= -\frac{24}{5} \end{aligned}$$

Miles's shadow is shrinking at the rate of $\frac{24}{5}$ feet per second throughout his walk. In particular, when he is twelve feet from the lamppost, his shadow is shrinking at the rate of $\frac{24}{5}$ feet per second.

(b) At what rate is the distance between the top of Miles's head and the top of the lamppost changing when Miles is twelve feet from the lamppost?

Solution: As above, let x be the distance from Miles to the lamppost. Let z be the distance from the top of Miles's head to the top of the lamppost. We know that the lamppost is five feet taller than Miles, so the distances x and z are the lengths of one leg and the hypotenuse of a right triangle whose third leg has length 5. We have $x^2 + 25 = z^2$.

We treat both x and z as functions of t and differentiate to get

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

We know that $\frac{dx}{dt} = -4$, and at the moment in question, we have $x = 12$. Using the fact that x and 5 are the legs of a right triangle with hypotenuse x, we find that $z = 13$. We get

$$2(12)(-4) = 2(13)\frac{dz}{dt}$$

which we solve for $\frac{dz}{dt}$, getting $\frac{dz}{dt} = -\frac{48}{13}$. The distance between the top of Miles's head and the top of the lamppost is decreasing at $\frac{48}{13}$ feet per second.

6. Find the following limits.

(a) $\displaystyle\lim_{x\to 0^-} \frac{e^x - 1}{x^2}$

Solution: The limit has the form $0/0$, so we may apply l'Hospital's rule to get

$$\lim_{x\to 0^-} \frac{e^x - 1}{x^2} = \lim_{x\to 0^-} \frac{e^x}{2x}$$

The new limit has form $1/0$, with the denominator approaching 0 through negative values. This tells us that the quotient goes to negative infinity as $x \to 0^-$. We conclude that

$$\lim_{x\to 0^-} \frac{e^x - 1}{x^2} = -\infty$$

(b) $\displaystyle\lim_{x\to 0^+} (\tan x)^x$

Solution: The limit is of the form 0^0, which is indeterminate. We write $y = \displaystyle\lim_{x\to 0^+} (\tan x)^x$ and take logarithms to get

$$\begin{aligned} \ln y &= \lim_{x\to 0^+} x \ln(\tan x) \\ &= \lim_{x\to 0^+} \frac{\ln(\tan x)}{1/x} \end{aligned}$$

which has form $-\infty/\infty$. By l'Hospital's rule, we have

$$\begin{aligned} \lim_{x\to 0^+} \frac{\ln(\tan x)}{1/x} &= \lim_{x\to 0^+} \frac{\frac{\sec^2 x}{\tan x}}{-1/x^2} \\ &= \lim_{x\to 0^+} \frac{-x^2}{\sin x \cos x} \end{aligned}$$

This still has an indeterminate form $(0/0)$, but we can split it up as

$$\begin{aligned} \lim_{x\to 0^+} \frac{-x^2}{\sin x \cos x} &= -\left(\lim_{x\to 0^+} \frac{x}{\sin x}\right)\left(\lim_{x\to 0^+} \frac{1}{\cos x}\right)\left(\lim_{x\to 0^+} x\right) \\ &= -(1)(1)(0) \\ &= 0. \end{aligned}$$

Since $\ln y = 0$, we conclude that $y = 1$.

7. Find the absolute maximum and minimum values of the function $f(x) = \dfrac{3x+4}{1+x^2}$ on the interval $[-2, 2]$.

Solution: To find the critical numbers of the given function, we compute

$$\begin{aligned} f'(x) &= \frac{3(1+x^2) - 2x(3x+4)}{(1+x^2)^2} \\ &= \frac{3 - 8x - 3x^2}{(1+x^2)^2} \\ &= \frac{(3+x)(1-3x)}{(1+x^2)^2} \end{aligned}$$

The derivative is zero at $x = -3$ and $x = \dfrac{1}{3}$. It is never undefined, since $1 + x^2$ is always greater than 0. The number $x = -3$ is outside the domain $[-2, 2]$, so we may ignore it. The number $x = \dfrac{1}{3}$ is in the domain, so that it a critical point. We evaluate f at the critical point and at the endpoints of the interval. We get

$$f(-2) = -\frac{2}{5}; \quad f\left(\frac{1}{3}\right) = \frac{9}{2}; \quad f(2) = 2$$

The absolute minimum value is $f(-2) = -\dfrac{2}{5}$ and the absolute maximum value is $f\left(\dfrac{1}{3}\right) = \dfrac{9}{2}$.