

Problem: Prove the quotient rule.

Suggested solution: Let f and g be functions. Then we have

$$\begin{aligned}\frac{d}{dx} \frac{f(x)}{g(x)} &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}\end{aligned}$$

We subtract and add $f(x)g(x)$ in the numerator to get

$$\begin{aligned}\frac{d}{dx} \frac{f(x)}{g(x)} &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{hg(x)g(x+h)} \\ &= \frac{g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}}{\lim_{h \rightarrow 0} g(x)g(x+h)} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.\end{aligned}$$