Math 203 Solution
Hour Exam I 9 – 10 October 2003

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1. Let P be the point $(3, -1, 5)$, Q be the point $(-2, 6, 2)$, and R be the point $(1, 4, -4)$.

   (a) Find the point on the x-axis closest to P, and find the distance from P to that point.

   Solution: The point on the x-axis closest to P is $(3, 0, 0)$, and the distance from P to $(3, 0, 0)$ is $\sqrt{26}$.

   (b) Find the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

   Solution: We have $\overrightarrow{PQ} = \langle -5, 7, -3 \rangle$ and $\overrightarrow{PR} = \langle -2, 5, -9 \rangle$, from which we get

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -48, -39, -11 \rangle.$$

   The area of the parallelogram is $\sqrt{48^2 + 39^2 + 11^3} = \sqrt{3946}$.

2. Let p_1 be the plane $4x - y + z = 0$. Suppose that p_1 intersects a plane p_2 along the line $\vec{r}(t) = 3t\vec{\imath} - t\vec{\jmath} + 4t\vec{k}$, and that p_1 and p_2 intersect at right angles.

   Find an equation for p_2.

   (HINT: A normal vector to p_2 lies in p_1 and is perpendicular to $\vec{r}(t)$.)

   Solution:

   The writer of the problem has made an error. The given line does not lie in plane p_1, so it can't be the line of intersection of p_1 and p_2.

   We can still find a plane that is perpendicular to p_1 and contains the given line. This seems about as close as we can come to an answer.

   Let $\vec{n}_1 = \langle 4, -1, 1\rangle$ be a normal vector to p_1. Since p_2 is perpendicular to p_1, it contains $\vec{n}_1$.

   Let $\vec{v} = \langle 3, -1, 4\rangle$ be a direction vector for the given line. The plane p_2 also contains $\vec{v}$.

   A normal vector to p_2 is given by

$$\vec{n}_1 \times \vec{v} \quad = \quad \langle -3, -13, -1\rangle.$$

   We know p_2 contains the origin, so we can immediately write down an equation for p_2:

$$3x + 13y + z = 0.$$

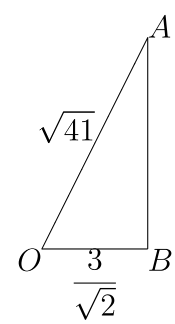
3. Let O denote the origin, A denote the point $(3,4,-4)$, and ℓ denote the line through the origin with direction vector $\vec{v}=\langle 1,-4,-1\rangle$.

(a) Find the scalar projection of the vector $\overrightarrow{OA}$ onto $\vec{v}$.
Solution: We use the formula for scalar projections, getting

$$\begin{aligned}\frac{\overrightarrow{OA}\cdot\vec{v}}{|\vec{v}|} &= \frac{\langle 3,4,-4\rangle\cdot\langle 1,-4,-1\rangle}{\sqrt{1^2+4^2+1^2}}\\ &= \frac{-9}{\sqrt{18}}\\ &= -\frac{3}{\sqrt{2}}\end{aligned}$$

(b) Let B be the point on ℓ closest to A. Find the lengths of line segments $\overline{OB}$ and $\overline{BA}$. (HINT: They are two legs of the right triangle OBA.)
Solution: We found the length of $\overline{OB}$ in part (a). It is $|-3/\sqrt{2}|=3/\sqrt{2}$.

We also know that the length of $\overline{OA}$ is $\sqrt{3^2+4^2+4^2}=\sqrt{41}$.
We can find the length of $\overline{BA}$ by using the Pythagorean theorem. We have



$$\begin{aligned}\overline{BA}^2 &= \overline{OA}^2-\overline{OB}^2\\ &= 41-\frac{9}{2}\\ &= \frac{73}{2}\end{aligned}$$

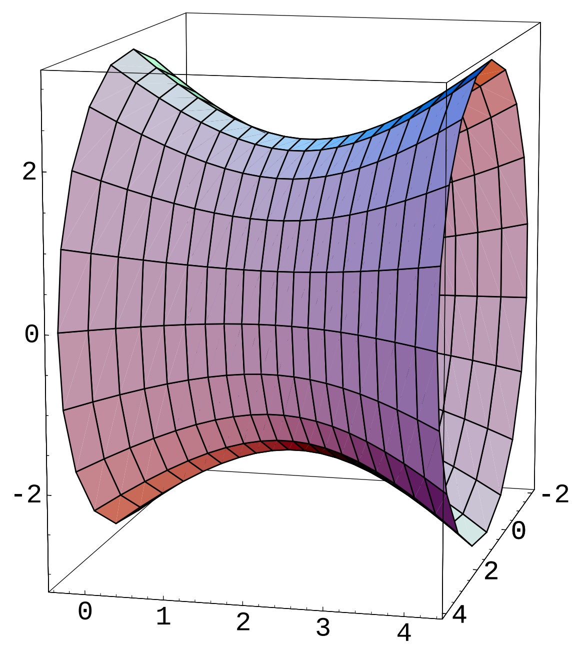
so that the length of $\overline{BA}$ is $\sqrt{\frac{73}{2}}$.

4. Identify and sketch the quadric surface $x^2 - 2x - y^2 + 4y + z^2 = 7$.

Solution: We rewrite the equation as

$$(x-1)^2 + z^2 = (y-2)^2 + 4$$

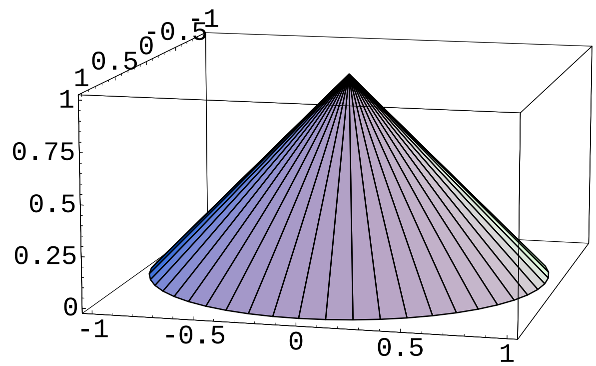
This is a (circular) hyperboloid of one sheet with axis parallel to the y-axis, center at $(1, 2, 0)$ and central radius 2. Here is a sketch:



5. Let S be the surface $z = 1 - \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

(a) Sketch S.

Solution: The surface is a conical cap. Here's a picture.



An easy way to see this is to convert the equation to cylindrical coordinates. We get $z = 1 - r$ or $r = 1 - z$. We can visualize the surface as being made up of circles about the z-axis whose radii decrease from 1 (at $z = 0$) to 0 (at $z = 1$).

(b) Find a equation for S in spherical coordinates. Try to make it as simple as possible. Remember that S isn't infinitely large, so you'll need to include limits on at least one coordinate.

Solution: Using the usual conversion formulas, we get

$$\begin{aligned} \rho \cos \phi &= 1 - \sqrt{\rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \sin^2 \phi \cos^2 \theta} \\ &= 1 - \sqrt{\rho^2 \sin^2 \phi} \\ &= 1 - \rho \sin \phi. \end{aligned}$$

We solve this for ρ, getting

$$\rho = \frac{1}{\sin \phi + \cos \phi},$$

with $0 \le \phi \le \frac{\pi}{2}$.

6. Let $f(x, y) = 2x^2 - 3xy + e^{1-x}y^3$.

(a) Find a linear approximation $L(x, y)$ for f at the point $(x, y) = (1, 2)$.

Solution: We have

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - 3y - e^{1-x}y^3 \\ \frac{\partial f}{\partial y} &= -3x + 3y^2 e^{1-x}. \end{aligned}$$

Evaluating these at the point $(1, 2)$, we get

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(1,2)} &= -10 \\ \left.\frac{\partial f}{\partial y}\right|_{(1,2)} &= 9. \end{aligned}$$

We have $f(1, 2) = 4$, so the linear approximation we want is

$$L(x, y) = 4 - 10(x - 1) + 9(y - 2).$$

(b) Suppose we want to keep the value of $f(x, y)$ approximately constant, equal to the value $f(1, 2)$. Due to circumstances beyond our control, the value of y suddenly jumps from 2 to 2.05. How should we adjust x to compensate?

Solution: The change from 2 to 2.05 in y causes the value of f to increase by

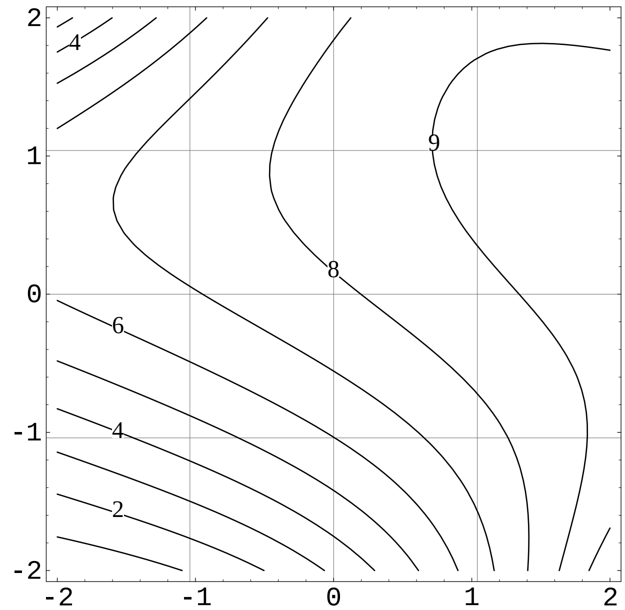
$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} \times 0.05 = 9 \times 0.05 = 0.45.$$

We need to find a Δx that will decrease f by 0.45. Since $\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = -10$, we need to solve

$$-10 \times \Delta x = -0.45$$

for Δx. We get $\Delta x = 0.045$. We should increase x to 1.045 to compensate for the increase in y.

7. Here is a contour plot of a function $f(x, y)$. As usual, the x-axis is horizontal and the y-axis is vertical.



Use the contour plot to estimate the following.

(a) $f(0, -1)$.
Solution: It looks like $f(0, -1) \approx 6$.

(b) $f_x(0, -1)$.
Solution: It looks like f increases by one unit as you move to the right by about $2/3$, so we guess that $f_x(0, -1) \approx \frac{3}{2}$.

(c) $f_y(0, -1)$
Solution: Between $(0, -1)$ and $(0, 0)$, the value of f increases by about 1.8. Between $(0, -2)$ and $(0, -1)$, the value of f increases by almost 3. Our two estimates for $f_y(0, -1)$ are 1.8 and 3. Let's take the average of these, and say that $f_y(0, -1) \approx 2.4$.

(d) The maximum and minimum values of f on the square $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.
Solution: The maximum value, somewhere between 10 and 11, occurs at $(2, -2)$. The minimum value, somewhere between 0 and 1, occurs at $(-2, -2)$.