Math 203 — Solution

Hour Exam II — 13 – 14 November 2003

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1. Let $f(x,y,z) = 2x^2y - 3yz$. Let $P$ be the point $(3,2,5)$. Find the directional derivative of $f$ at $P$ in the direction from $P$ toward the $z$-axis.

   Solution: We have

   $$\nabla f = 4xy\vec{\imath} + (2x^2 - 3z)\vec{\jmath} - 3y\vec{k}$$

   so that

   $$\nabla f(3,2,5) = 24\vec{\imath} + 3\vec{\jmath} - 6\vec{k}.$$

   The direction from the point $(3,2,5)$ toward the $z$-axis is $-3\vec{\imath} - 2\vec{\jmath}$. A unit vector in this direction is

   $$\vec{u} = -\frac{3}{\sqrt{13}}\vec{\imath} - \frac{2}{\sqrt{13}}\vec{\jmath}.$$

   The directional derivative we want is the dot product

   $$\begin{aligned} \nabla f(3,2,5) \cdot \vec{u} &= -\frac{3}{\sqrt{13}} \cdot 24 - \frac{2}{\sqrt{13}} \cdot 3 \\ &= -\frac{78}{\sqrt{13}} \\ &= -6\sqrt{13} \end{aligned}$$

2. Suppose $f(s,t) = \sqrt{s}\sin t$ and that $s = 1 + t^2$.

   (a) Find $\dfrac{\partial f}{\partial t}$ at $t = \dfrac{\pi}{2}$.
   Solution: We have

   $$\frac{\partial f}{\partial t} = \sqrt{s}\cos t$$

   When $t = \dfrac{\pi}{2}$, we get $s = 1 + \dfrac{\pi^2}{4}$. Evaluating the partial derivative at $s = 1 + \dfrac{\pi^2}{4}$, $t = \dfrac{\pi}{2}$ gives

   $$\left.\frac{\partial f}{\partial t}\right|_{t=\frac{\pi}{2}} = 0$$

(b) Find $\dfrac{df}{dt}$ at $t = \dfrac{\pi}{2}$.
Solution: We have

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial s}\frac{ds}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt} \\ &= \frac{1}{2\sqrt{s}} \sin t (2t) + \sqrt{s}\cos t \\ &= \frac{t}{\sqrt{s}} \sin t + \sqrt{s}\cos t \end{aligned}$$

As before, the point at which we want to evaluate all these derivatives is $s = 1+\dfrac{\pi^2}{4}$, $t = \dfrac{\pi}{2}$. We get

$$\begin{aligned} \left.\frac{df}{dt}\right|_{t=\frac{\pi}{2}} &= \frac{\frac{\pi}{2}}{\sqrt{1+\frac{\pi^2}{4}}} \\ &= \frac{\pi}{\sqrt{4+\pi^2}} \end{aligned}$$

3. Find all the critical points of the function $f(x,y) = x^2y^2 - 9y^2 - 2x^2 - 6x$. Do not bother to classify them.

   Solution: We have

$$\begin{aligned} f_x(x,y) &= 2xy^2 - 4x - 6 \\ f_y(x,y) &= 2x^2y - 18y. \end{aligned}$$

   Setting $f_y(x,y) = 0$, we get

$$0 = 2y(x^2 - 9)$$

   so either $y = 0$ or $x = \pm 3$.

   If $y = 0$, the equation $f_x(x,y) = 0$ becomes $4x + 6 = 0$, so $x = -\dfrac{3}{2}$.

   If $x = 3$, the equation $f_x(x,y) = 0$ says

$$0 = 6y^2 - 12 - 6$$

   so that $y^2 = 3$ and $y = \pm\sqrt{3}$.

If $x = -3$, the equation $f_x(x, y) = 0$ says

$$0 = -6y^2 + 12 - 6$$

so $y^2 = 1$ and $y = \pm 1$.

There are five critical points:

$$\left(-\frac{3}{2}, 0\right), \ (3, \sqrt{3}), \ (3, -\sqrt{3}), \ (-3, 1), \ \text{and} \ (-3, -1).$$

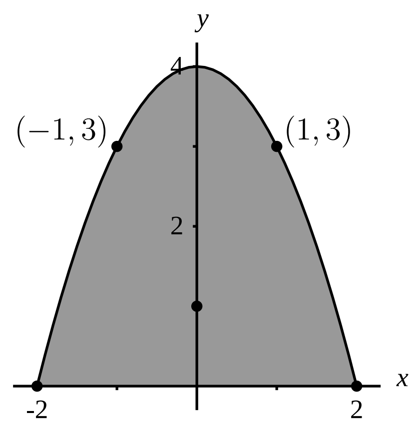
4. Let $R$ be the region in the $xy$-plane bounded by the $x$-axis and the parabola $y = 4 - x^2$. Find the absolute maximum and minimum values of the function $f(x, y) = xy - x$ on the region $R$.

Solution: First we identify any critical points of $f$ inside the region. We have

$$\begin{aligned} f_x(x, y) &= y - 1 \\ f_y(x, y) &= x \end{aligned}$$

Setting these equal to zero, we find that $(0, 1)$ is a critical point of $f$. Since $(0, 1)$ lies inside $R$, we evaluate $f$ at $(0, 1)$ and begin our list of possible extrema with

$$f(0, 1) = 0.$$




Along the bottom edge of $R$, we have $y = 0$, so $f(x, y) = f(x, 0) = -x$. Clearly, the maximum and minimum values of $f$ along the bottom edge occur at the endpoints, $x = -2$ and $x = 2$. We have

$$\begin{aligned} f(-2, 0) &= 2 \\ f(2, 0) &= -2. \end{aligned}$$

Along the top edge, we have $y = 4 - x^2$, so that

$$\begin{aligned} f(x, y) &= f(x, 4 - x^2) \\ &= x(4 - x^2) - x \\ &= 4x - x^3 - x \\ &= 3x - x^3. \end{aligned}$$

We differentiate with respect to $x$ to get

$$\begin{aligned} \frac{d}{dx} f(x, 4-x^2) &= 3 - 3x^2 \\ &= 3(1-x^2). \end{aligned}$$

The critical points along this arc are at $x = \pm 1$. (There are also critical points at the endpoints of the arc, but we've already listed those.)

When $x = \pm 1$, we get $y = 3$, so we complete our list of possible extrema with

$$\begin{aligned} f(-1, 3) &= -2 \\ f(1, 3) &= 2. \end{aligned}$$

Among all the points we've listed, the maximum value is $f(1, 3) = f(-2, 0) = 2$ and the minimum value is $f(-1, 3) = f(2, 0) = -2$.

5. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x^2 + x + y^2 - 3y$ on the circle $x^2 + y^2 = 10$.

   Solution: Let $g(x, y) = x^2 + y^2$. Then we have

$$\begin{aligned} \nabla f &= \langle 2x + 1, 2y - 3 \rangle \\ \nabla g &= \langle 2x, 2y \rangle \end{aligned}$$

   and our system of equations is

$$\begin{aligned} 2x + 1 &= 2\lambda x \\ 2y - 3 &= 2\lambda y \\ x^2 + y^2 &= 10 \end{aligned}$$

   We multiply the first equation by $y$ and the second by $x$ to get

$$\begin{aligned} 2xy + y &= 2\lambda xy \\ 2xy - 3x &= 2\lambda xy \end{aligned}$$

   from which we conclude that $y = -3x$. Substituting $y = -3x$ into the third equation above, we get

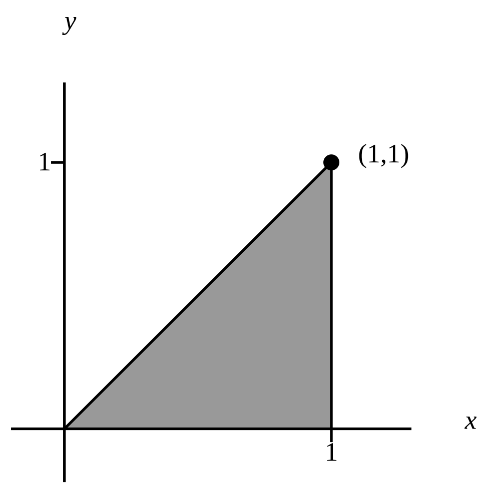
$$x^2 + 9x^2 = 10$$

so that $x = \pm 1$. The points where the minimum and maximum values of $f$ could occur are $(1, -3)$ and $(-1, 3)$. We have

$$\begin{aligned} f(1,-3) &= 1+1+9+9 \\ &= 20 \\ f(-1,3) &= 1-1+9-9 \\ &= 0 \end{aligned}$$

The maximum value of $f$ on the circle is 20, and the minimum value is 0.

6. Let $R$ be the region in the $xy$-plane shown at right. Set up (but do not evaluate) $\iint\limits_R \sqrt{x^2+y^2}\,dA$ in three ways:




(a) In rectangular coordinates, as an integral $dy\,dx$.
Solution: We have

$$\iint\limits_R \sqrt{x^2+y^2}\,dA = \int_0^1 \int_0^x \sqrt{x^2+y^2}\,dy\,dx$$

(b) In rectangular coordinates, as an integral $dx\,dy$.
Solution: We have

$$\iint\limits_R \sqrt{x^2+y^2}\,dA = \int_0^1 \int_y^1 \sqrt{x^2+y^2}\,dx\,dy$$

(c) In polar coordinates.

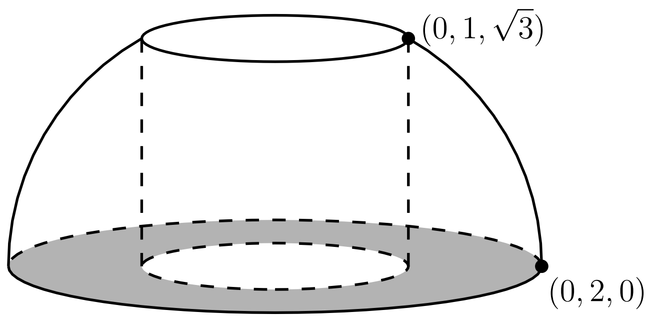
Solution: The line $x = 1$ can be expressed in polar coordinates as $r\cos\theta = 1$, or $r = \sec\theta$. We have

$$\iint_R \sqrt{x^2+y^2}\,dA \quad = \quad \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} (r)r\,dr\,d\theta$$

7. Let $E = \{(x, y, z) \,:\, z \geq 0,\ x^2 + y^2 + z^2 \leq 4,\ \text{ and } x^2 + y^2 \geq 1\}$. Set up (but do not evaluate) $\iiint_E x^2\,dV$ in two ways:

(a) In cylindrical coordinates.

Solution: The region $E$ is the upper hemisphere of the ball centered at the origin with radius 2, with a cylindrical hole drilled out. Here is a picture:




In cylindrical coordinates, the integral is

$$\int_0^{2\pi} \int_1^2 \int_0^{\sqrt{4-r^2}} (r\cos\theta)^2\,r\,dz\,dr\,d\theta$$

(b) In spherical coordinates.

Solution: The trick here is to figure out how the cylinder $x^2 + y^2 = 1$ should be used to set a bound on $\rho$. We know that

$$\begin{aligned} x^2 + y^2 &= \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta \\ &= \rho^2\sin^2\varphi \end{aligned}$$

In spherical coordinates, the cylinder has equation $\rho^2\sin^2\varphi = 1$. Since both $\rho$ and $\sin\varphi$ are positive in the region in question, this says that $\rho\sin\varphi = 1$, or $\rho = \csc\varphi$ along the cylinder wall.

The lower limit for $\varphi$ is determined by the fact that the top edge of $E$ has height $\sqrt{3}$ and radius 1.

The spherical-coordinates integral is

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\csc \varphi}^{2} (\rho \sin \varphi \cos \theta)^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$