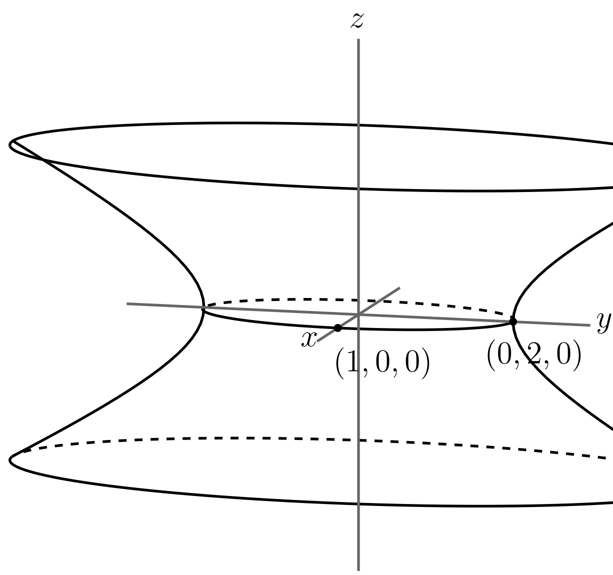
Math 203(01) **Final Exam II Solution** **December 2003**

1. Let $f(x,y,z) = x^2 + \dfrac{y^2}{4} - z^2$.

   (a) Identify and sketch the surface $f(x,y,z) = 1$.

   Solution: The equation $x^2 + \dfrac{y^2}{4} - z^2 = 1$ describes an elliptic hyperboloid of one sheet about the z-axis. Here is a sketch:



   (b) Find an equation for the plane tangent to the surface $f(x,y,z) = 1$ at the point $(1,2,1)$.

   Solution: The vector $\nabla f(1,2,1)$ is normal to the plane we want. We compute

$$\nabla f(x,y,z) \quad = \quad 2x\vec{\imath} + \frac{y}{2}\vec{\jmath} - 2z\vec{k}$$

   so that

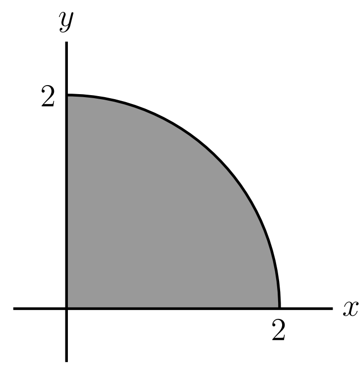
$$\nabla f(1,2,1) \quad = \quad 2\vec{\imath} + \vec{\jmath} - 2\vec{k}$$

   An equation for the tangent plane is

$$2(x-1) + (y-2) - 2(z-1) = 0$$

2. Compute $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$.

Solution: The region over which this integral is taken, shown at right, is most conveniently described in polar coordinates as

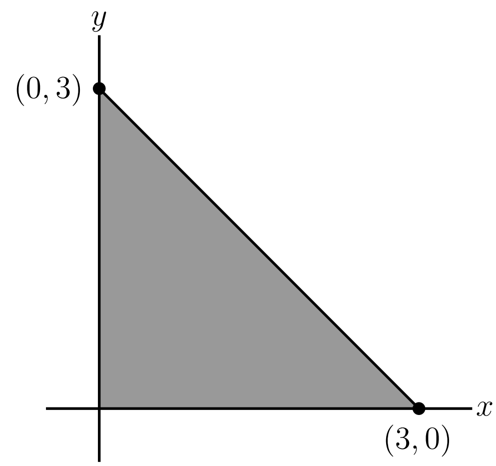


$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2$$

The integral becomes

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_0^2 e^{-r^2} r\, dr\, d\theta &= \frac{\pi}{2}\left[-\frac{e^{-r^2}}{2}\right]_0^2 \\
&= \frac{\pi}{4}\left[-e^{-4}+1\right] \\
&= \frac{\pi}{4}(1-e^{-4})
\end{aligned}$$

3. Let $f(x, y) = 2x^2 - 12x + y^2 - 2y$. Find the maximum and minimum values of f on the shaded region shown at right.



Solution: We first compute

$$f_x = 4x - 12 \text{ and } f_y = 2y - 2$$

to find that the function f has a single critical point at $(3, 1)$. This point is not in the shaded region, so we ignore it.

Along the left side of the region, $x = 0$, and we have

$$f(0, y) \quad = \quad y^2 - 2y$$

We take the derivative to find $f'(0, y) = 2y - 2$. There is a critical number at $y = 1$. If there is an extremum of f along the left side of the region, it must be at $(0, 0)$, $(0, 1)$, or $(0, 3)$. We compute

$$f(0, 0) = 0, \quad f(0, 1) = -1, \quad f(0, 3) = 3$$

Along the bottom side of the region, $y = 0$, and we have

$$f(x, 0) \quad = \quad 2x^2 - 12x$$

We take the derivative to find $f'(x, 0) = 4x - 12$. There is a critical number at $x = 3$. If there is an extremum of f along the bottom side of the region, it must be at $(0, 0)$ or $(3, 0)$. We already know that $f(0, 0) = 0$. We find that $f(3, 0) = -18$.

Along the right side of the region, $y = 3 - x$, and we have

$$\begin{aligned} f(x, 3 - x) &= 2x^2 - 12x + (3 - x)^2 - 2(3 - x) \\ &= 3x^2 - 16x + 3 \end{aligned}$$

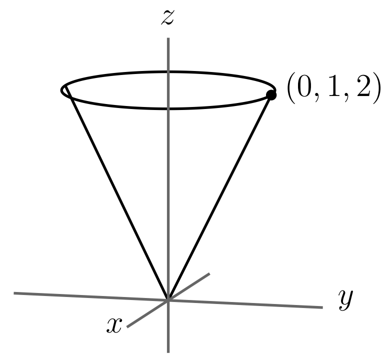
We find that $f'(x) = 6x - 16$, so there is a critical number at $x = \frac{8}{3}$. If there is an extremum of f on the right side of the region, it must be at one of the endpoints (where we already know the values of f) or at the point $\left(\frac{8}{3}, \frac{1}{3}\right)$. We find that

$$f\left(\frac{8}{3}, \frac{1}{3}\right) = -\frac{55}{3}$$

The least of all the values we have found is $-\frac{55}{3}$, and the greatest is 3.

4. Let S be the part of the cone $z^2 = 4x^2 + 4y^2$ between the planes $z = 0$ and $z = 2$. Compute $\displaystyle\iint_S z\,dS$.

Solution: We can rewrite the equation of the cone as



$$z = \sqrt{4(x^2 + y^2)} = 2r$$

Using this, we view the cone as a stack of circles, each at a height equal to twice its radius. We parametrize the cone as

$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, 2u\rangle$$

with $0 \le u \le 1$ and $0 \le v \le 2\pi$. From this, we find

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \langle \cos v, \sin v, 2\rangle \times \langle -u\sin v, u\cos v, 0\rangle \\ &= \langle -2u\cos v, -2u\sin v, u\rangle\end{aligned}$$

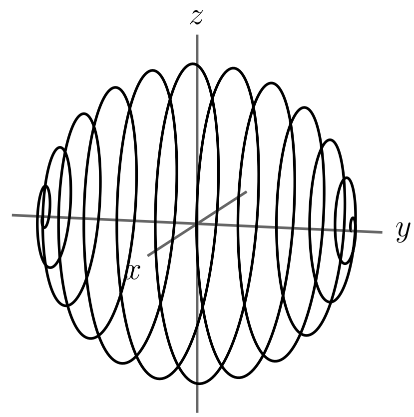
so that $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4u^2 + u^2} = \sqrt{5}u$.
Using $z = 2u$ from the parametrization, we get

$$\begin{aligned}\iint_S z\,dS &= \int_0^{2\pi}\int_0^1 (2u)\sqrt{5}u\,du\,dv \\ &= 2\pi\left[2\sqrt{5}\frac{u^3}{3}\right]_0^1 \\ &= \frac{4\pi\sqrt{5}}{3}\end{aligned}$$

5. Let C be the curve parametrized by

$\mathbf{r}(t) = \sin(t)\cos(24t)\vec{\imath} + \cos(t)\vec{\jmath} + \sin(t)\sin(24t)\vec{k}$

with $0 \leq t \leq \pi$. The curve C is shown in the diagram at right.



Let

$\mathbf{F}(x, y, z) = 2xy^3\vec{\imath} + (3e^z + 3x^2y^2)\vec{\jmath} + 3ye^z\vec{k}$

Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: We note that $\mathbf{F}$ is the gradient of the function

$$f(x, y, z) = x^2y^3 + 3ye^z$$

so that $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{end of } C) - f(\text{beginning of } C)$.

The end of C is $\mathbf{r}(\pi) = (0, -1, 0)$, and the beginning of C is $\mathbf{r}(0) = (0, 1, 0)$. Thus

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(0, -1, 0) - f(0, 1, 0) \\ &= -3 - (3) \\ &= -6 \end{aligned}$$

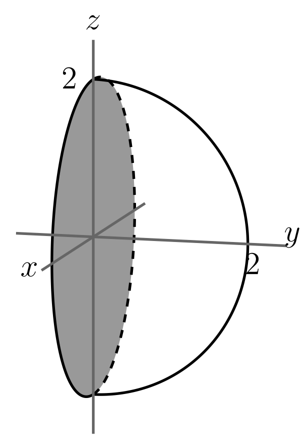
6. Let S_1 be the hemispherical surface $x^2 + y^2 + z^2 = 4$, $y \geq 0$, oriented in the direction of the positive y-axis. Let S_2 be the disk $x^2 + z^2 \leq 4$, $y = 0$, oriented in the direction of the negative y-axis.

   Let S be the closed surface formed by joining S_1 and S_2. That is, let $S = S_1 \cup S_2$.

   Let $\mathbf{F}(x, y, z) = \left\langle x + yz^2, y - 4, x^3y^3 - z \right\rangle$.

   (a) Compute $\displaystyle\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

We parametrize S_2 (the shaded surface in the picture) as



$$\mathbf{r}(u, v) \;=\; \langle u \cos v, 0, u \sin v \rangle$$

with $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$. We get

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \langle \cos v, 0, \sin v \rangle \times \langle -u \sin v, 0, u \cos v \rangle \\ &= \langle 0, -u, 0 \rangle \end{aligned}$$

The negative sign on the u says that we have the correct orientation.
We need to find only the $\vec{\jmath}$ component of $\mathbf{F}(\mathbf{r}(u, v))$. We get

$$\mathbf{F}(\mathbf{r}(u, v)) \;=\; \langle -, 0 - 4, - \rangle$$

so that

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^2 \langle -, -4, - \rangle \cdot \langle 0, -u, 0 \rangle \, du \, dv \\ &= \int_0^{2\pi} \int_0^2 4u \, du \, dv \\ &= 2\pi [2u^2]_0^2 \\ &= 16\pi \end{aligned}$$

Alternatively, we could observe that the unit normal to S_2 is $-\vec{\jmath}$, and that the $\vec{\jmath}$ component of $\mathbf{F}$ along S_2 is constantly equal to -4, so that $\iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}$ is simply 4 times the area of S_2. We get

$$\begin{aligned}\iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S} &= 4 \cdot \pi(2)^2 \\ &= 16\pi\end{aligned}$$

(b) Compute $\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S}$. (HINT: Think "divergence theorem.")

We get

$$\text{div } \mathbf{F} = 1 + 1 - 1 = 1$$

By the divergence theorem,

$$\begin{aligned}\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} &= \iiint\limits_{E} 1\, dV \\ &= \text{Volume of } E\end{aligned}$$

where E is the solid enclosed by S. The solid E is a hemisphere of radius 2, so its volume is

$$\frac{2}{3}\pi(2)^3 = \frac{16\pi}{3}$$

(c) Compute $\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

We have

$$\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

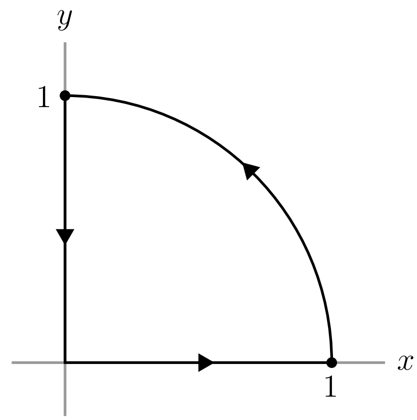
Using the results above, this says that

$$\begin{aligned}\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \frac{16\pi}{3} - 16\pi \\ &= -\frac{32\pi}{3}\end{aligned}$$

7. Let C be the closed path in the xy-plane shown in the picture at right, oriented counterclockwise seen from above. (The curved part of C is one quarter of the circle $x^2 + y^2 = 1$, $z = 0$.)
Let

$$\mathbf{F}(x, y, z) = (x^2 - y^2)\vec{\imath} + (z\cos(y) + x)\vec{\jmath} + xy^2\vec{k}$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.



Solution: By Stokes's theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where S is any capping surface for C. One such surface is the quarter-disk $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 1$ in the xy-plane. The unit normal to this surface is $\vec{k}$, so we can compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ by integrating the $\vec{k}$ component of curl $\mathbf{F}$ over the region $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 1$.

We find the $\vec{k}$ component of curl $\mathbf{F}$ as

$$\frac{\partial}{\partial x}(z\cos y + x) - \frac{\partial}{\partial y}(x^2 - y^2) = 1 + 2y$$

Thus we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_S 1 + 2y\, dA \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 (1 + 2r\sin\theta) r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta \right]_0^1 d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{2}{3}\sin\theta\, d\theta \\
&= \frac{\pi}{4} + \frac{2}{3}
\end{aligned}$$