Math 203 Practice Exam I – Solution October 2003

1. Let P be the point $(0, 2, 1)$, and Q be the point $(-1, 3, 5)$.

   (a) Write a vector parametrization of the line segment $\overline{PQ}$.
   Solution: The vector $\overrightarrow{PQ}$ is $\langle -1, 1, 4\rangle$, so the parametrization we want is

$$\vec{r}(t) \quad = \quad -t\vec{\imath} + (2+t)\vec{\jmath} + (1+4t)\vec{k}, \quad 0 \le t \le 1.$$

   (b) Find the area and the perimeter of triangle whose vertices are P, Q, and the origin.
   Solution: To begin, we find the cross product of $\langle 0, 2, 1\rangle$ and $\langle -1, 3, 5\rangle$. We get $\langle 7, -1, 2\rangle$. The area of the triangle is one half the magnitude of this cross product, that is $\dfrac{\sqrt{54}}{2}$.
   To find the perimeter, we need the lengths of the sides. Denoting the origin by O, we have $|OP| = \sqrt{5}$, $|OQ| = \sqrt{35}$ and $|PQ| = \sqrt{18}$. The perimeter of the triangle is $\sqrt{5} + \sqrt{35} + \sqrt{18}$.

2. Let p_1 be the plane $2x + 4y - z = 5$ and p_2 be the plane $x - 3y - 2z = 2$.

(a) Find a vector parametrization for the line of intersection of p_1 and p_2.

Solution: Let $\vec{n}_1$ and $\vec{n}_2$ be normal vectors to planes p_1 and p_2 respectively. We may take $\vec{n}_1 = \langle 2, 4, -1\rangle$ and $\vec{n}_2 = \langle 1, -3, -2\rangle$.

The line of intersection of p_1 and p_2 must be perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, so its direction is given by

$$\vec{n}_1 \times \vec{n}_2 = \langle -11, 3, -10\rangle.$$

To find a point on the line of intersection, we need to find (x, y, z) satisfying the equations $2x + 4y - z = 5$ and $x - 3y - 2z = 2$. We arbitrarily set $y = 0$ to get the system

$$\begin{array}{rcrcl} 2x & - & z & = & 5 \\ x & - & 2z & = & 2 \end{array}$$

The solution is $x = \dfrac{8}{3}$, $z = \dfrac{1}{3}$. The point $\left(\dfrac{8}{3}, 0, \dfrac{1}{3}\right)$ lies on the line, and we can write a vector parametrization of the intersection line as

$$\vec{r}(t) = \left(\frac{8}{3} - 11t\right)\vec{\imath} + 3t\vec{\jmath} + \left(\frac{1}{3} - 10t\right)\vec{k}.$$

(b) Find the cosine of the angle of intersection of the planes p_1 and p_2.

Solution: Let θ denote the angle of intersection. We have

$$\begin{aligned} \cos\theta &= \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} \\ &= -\frac{8}{\sqrt{21}\sqrt{14}} \\ &= -\frac{8}{7\sqrt{6}}. \end{aligned}$$

3. Let ℓ be the line with parametric equations

$$\begin{aligned} x &= 2t \\ y &= -2t \\ z &= -t \end{aligned}$$

and let $\vec{v} = \langle 2, -5, 6 \rangle$. Find vectors $\vec{a}$ and $\vec{b}$ so that $\vec{a}$ is parallel to ℓ, $\vec{b}$ is perpendicular to ℓ, and $\vec{a} + \vec{b} = \vec{v}$.

Solution: The vector $\vec{a}$ that we want is just the vector projection of $\vec{v}$ onto the direction vector of ℓ, which is $\langle 2, -2, -1 \rangle$. We have

$$\begin{aligned} \vec{a} &= (\vec{v} \cdot \langle 2, -2, -1 \rangle) \frac{\langle 2, -2, -1 \rangle}{9} \\ &= \frac{8}{9} \langle 2, -2, -1 \rangle \\ &= \left\langle \frac{16}{9}, -\frac{16}{9}, -\frac{8}{9} \right\rangle . \end{aligned}$$

Then $\vec{b}$ is simply $\vec{v} - \vec{a}$. We get

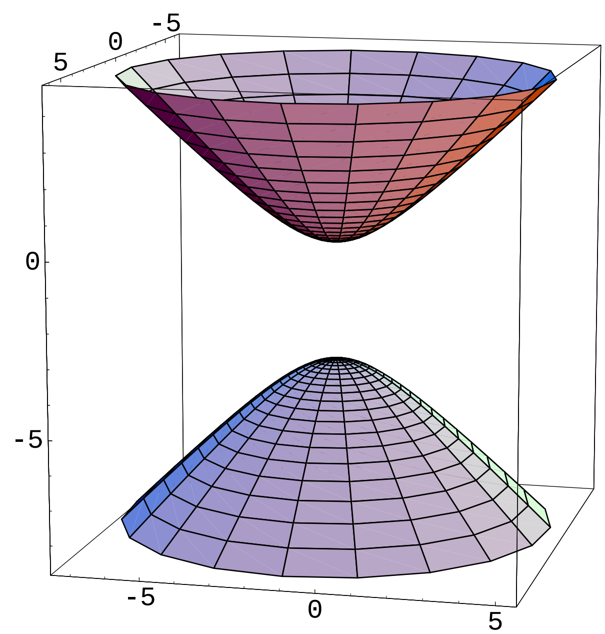
$$\begin{aligned} \vec{b} &= \langle 2, -5, 6 \rangle - \left\langle \frac{16}{9}, -\frac{16}{9}, -\frac{8}{9} \right\rangle \\ &= \left\langle \frac{2}{9}, -\frac{29}{9}, \frac{62}{9} \right\rangle . \end{aligned}$$

4. Identify and sketch the quadric surface $x^2 + y^2 + 2y - z^2 - 4z = 0$.

Solution: We rewrite the equation as

$$x^2 + (y+1)^2 = (z+2)^2 - 3.$$

This is a (circular) hyperboloid of two sheets with axis parallel to the z-axis, center at $(0, -1, -2)$, and vertices at $-2 \pm \sqrt{3}$. Here is a sketch:



5. (a) Find the cylindrical and rectangular coordinates for the point whose spherical coordinates are $\rho = 3$, $\theta = \dfrac{\pi}{4}$, $\varphi = \dfrac{\pi}{6}$.

Solution: The cylindrical coordinates are

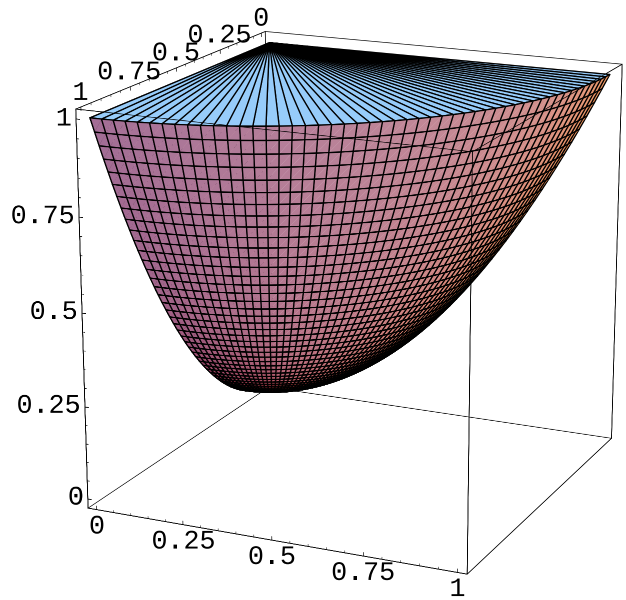
$$\begin{aligned} r &= \rho \sin\varphi \\ &= \frac{3}{2} \\ z &= \rho\cos\varphi \\ &= \frac{3\sqrt{3}}{2} \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

The rectangular coordinates are

$$\begin{aligned} x &= r\cos\theta \\ &= \frac{3}{2}\frac{\sqrt{2}}{2} \\ &= \frac{3\sqrt{2}}{4} \\ y &= r\sin\theta \\ &= \frac{3}{2}\frac{\sqrt{2}}{2} \\ &= \frac{3\sqrt{2}}{4} \\ z &= \rho\cos\varphi \\ &= \frac{3\sqrt{3}}{2} \end{aligned}$$

(b) Sketch the region described by $0 \leq \theta \leq \dfrac{\pi}{2}$, $0 \leq r \leq 1$, $r^2 \leq z \leq 1$.

Solution: This is the region in the first octant above the paraboloid $z = r^2$ and below the plane $z = 1$. Here is a sketch.



0
0.25
0.5
0.75
1
1
0.75
0.5
0.25
0
0
0.25
0.5
0.75
1

6. My motorcycle gets about 50 miles to the gallon. I use the trip odometer to keep track of how far I've ridden between refuelings. When the trip odometer reads about 100 miles, I stop for fuel, and the tank usually takes about 2 gallons.

   Of course, the odometer doesn't always read exactly 100 miles at each fuel stop (there might not be a gas station right at that place in the road), and the tank doesn't always take exactly 2 gallons.

   Let $f(m,g)$ denote the motorcycle's fuel economy (in miles per gallon) as a function of m, the number of miles driven, and g the number of gallons of fuel used. Find a linear function $L(m,g)$ that best approximates f when m is near 100 and g is near 2.

   Solution: First we need the formula for f, which is clearly $f(m,g) = \dfrac{m}{g}$. From this we get

$$\begin{aligned} \frac{\partial f}{\partial m} &= \frac{1}{g}, \\ \frac{\partial f}{\partial g} &= -\frac{m}{g^2}. \end{aligned}$$

   We evaluate these partials at the point in question to get

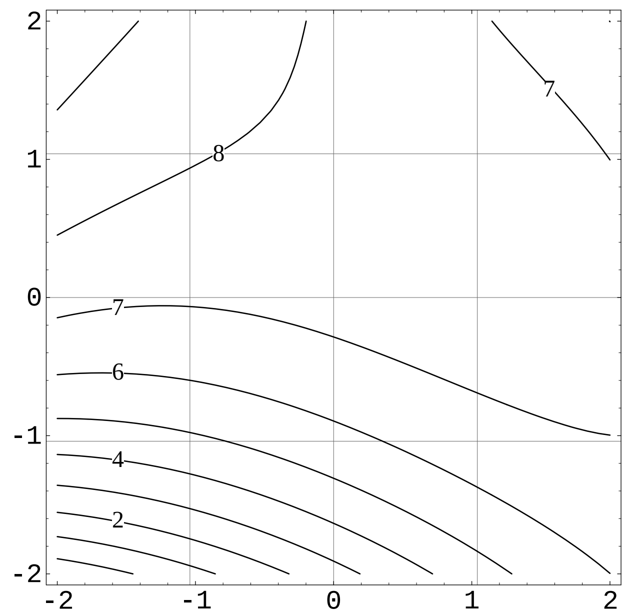
$$\begin{aligned} \left.\frac{\partial f}{\partial m}\right|_{(100,2)} &= \frac{1}{2} \\ \left.\frac{\partial f}{\partial g}\right|_{(100,2)} &= -25 \end{aligned}$$

   The linear function we want is

$$L(m,g) = 50 + \frac{1}{2}(m-100) - 25(g-2)$$

   For each mile in excess of 100, we add $\frac{1}{2}$ mpg to our economy estimate; for each, say, tenth of a gallon in excess of 2.0 gallons, we subtract 2.5 mpg from our estimate.

7. Here is a contour plot of a function $f(x, y)$. Use the contour plot to construct a reasonable linear approximation to f at the point $(-1, -1)$.



Solution: From the picture, we see that $f(-1, -1)$ is somewhere between 4 and 5, and probably closer to 5. We estimate $f(-1, -1)$ as 4.8. The value of f_x at $(-1, -1)$ is positive, and we can estimate it by looking at the rate of change in f as we move to the right and left along the line $y = -1$. We guess that $f_x(-1, -1)$ is a little less than 1. Let's call it 0.8.

We can estimate $f_y(-1, -1)$ by looking at the rate of change in f as we move up and down along the line $x = -1$. Going upward from $(-1, -1)$, we see a rate of change that's probably between 2 and 3; going downward we find a rate of change that's between 3 and 4. Let's guess that $f_y(-1, -1)$ is about 3.

Our linear approximation is then

$$L(x, y) \quad = \quad 4.8 + 0.8(x + 1) + 3(y + 1).$$