Math 203 Solution
Practice Exam II 10 November 2003

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1. Let $f(x, y, z) = 3x^2z - 2y^2$. Let P be the point $(4, 1, 2)$. Find the directional derivative of f at P in the direction from P toward the x-axis.

   Solution: We have

$$\nabla f = 6xz\vec{\imath} - 4y\vec{\jmath} + 3x^2\vec{k}$$

   so that

$$\nabla f(4, 1, 2) = 48\vec{\imath} - 4\vec{\jmath} + 48\vec{k}.$$

   The direction from the point $(4, 1, 2)$ toward the x-axis is $-\vec{\jmath} - 2\vec{k}$. A unit vector in this direction is

$$\vec{u} = -\frac{1}{\sqrt{5}}\vec{\jmath} - \frac{2}{\sqrt{5}}\vec{k}.$$

   The directional derivative we want is the dot product

$$\begin{aligned} \nabla f(4, 1, 2) \cdot \vec{u} &= -\frac{1}{\sqrt{5}} \cdot (-4) - \frac{2}{\sqrt{5}} \cdot (48) \\ &= -\frac{92}{\sqrt{5}}. \end{aligned}$$

2. Sand runs into the lower globe of an hourglass at 2 cm^3/min and forms a pile in the shape of a right circular cone. After about two minutes and twenty seconds, the height of the pile reaches 2 cm and the radius (of the base) reaches 1.5 cm. Suppose that the height of the pile is increasing at 0.2 cm/sec at that moment. How fast is the radius increasing at the same moment?

Solution: We recall that the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height. We view r and h as functions of t, and differentiate using the chain rule to get

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ &= \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}. \end{aligned}$$

At the moment in question, we know $\frac{dV}{dt} = 2$, $\frac{dh}{dt} = 0.2$, $r = 1.5$, and $h = 2$. We get

$$2 = 2\pi \frac{dr}{dt} + \frac{3}{4}\pi(0.2),$$

which we can easily solve for $\frac{dr}{dt}$. We get

$$\frac{dr}{dt} = \frac{40 - 3\pi}{40\pi} \frac{\text{cm}}{\text{min}}.$$

3. Find and classify all the critical points of the function $f(x,y) = x^2y^2 - 4x^2 - y^2 + 12y$.

Solution: We have

$$\begin{aligned} f_x(x,y) &= 2xy^2 - 8x \\ f_y(x,y) &= 2x^2y - 2y + 12. \end{aligned}$$

Setting $f_x(x,y) = 0$, we get

$$0 = 2x(y^2 - 4)$$

so that either $x = 0$ or $y = \pm 2$.

If $x = 0$, then the equation $f_y(x,y) = 0$ says $-2y + 12 = 0$, so $y = 6$.

If $y = 2$, the equation $f_y(x,y) = 0$ says

$$\begin{aligned} 0 &= 4x^2 - 4 + 12 \\ &= 4x^2 + 8 \end{aligned}$$

Since $x^2 \geq 0$ for all real x, there are no solutions to this equation, and thus no critical points with $y = 2$.

If $y = -2$, the equation $f_y(x,y) = 0$ says

$$\begin{aligned} 0 &= -4x^2 + 4 + 12 \\ &= -4x^2 + 16 \end{aligned}$$

so that $x^2 = 4$ and $x = \pm 2$.

There are three critical points altogether:

$$(0,6),\ (2,-2),\ \text{and } (-2,-2)$$

To classify them, we find

$$f_{xx}(x,y) = 2y^2 - 8; \quad f_{xy}(x,y) = 4xy; \quad f_{yy}(x,y) = 2x^2 - 2$$

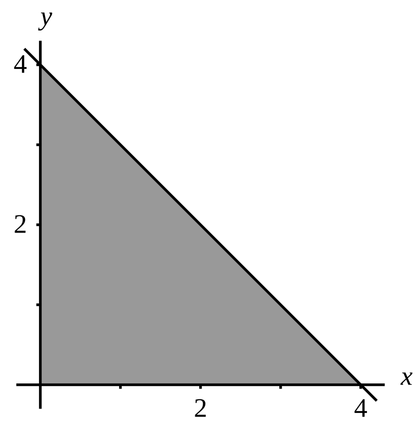
so that $D(x,y) = (2y^2 - 8)(2x^2 - 2) - 16x^2y^2$.

We find that $D(0,6) = 64(-2) < 0$, so $(0,6)$ is a saddle point.

At $x = \pm 2$, $y = -2$, we get $D(\pm 2, -2) = 0 - 256 < 0$, so these are saddle points as well.

4. Let R be the region in the xy-plane with $x \geq 0$, $y \geq 0$, and $x + y \leq 4$. Find the minimum and maximum values of the function $f(x, y) = x^2 - 2x + y^2 - 4y$ on the region R.

Solution: First we identify any critical points in the interior of R. We have



$$\begin{aligned} f_x(x, y) &= 2x - 2 \\ f_y(x, y) &= 2y - 4 \end{aligned}$$

The function has a single critical point at $(1, 2)$. Since this point lies inside R, we list it as a possible extremum. We have

$$\begin{aligned} f(1, 2) &= 1 - 2 + 4 - 8 \\ &= -5 \end{aligned}$$

Along the bottom edge of R, we have $y = 0$, so we consider

$$g(x) = f(x, 0) = x^2 - 2x$$

We have $g'(x) = 2x - 2$, so the only critical number for g is $x = 1$. Thus f may have an extremum at $(1, 0)$. We add this to our list of possible extrema, along with $(0, 0)$ and $(4, 0)$, the endpoints of the bottom edge of R. We have

$$\begin{aligned} f(1, 0) &= -1 \\ f(0, 0) &= 0 \\ f(4, 0) &= 8 \end{aligned}$$

Along the left edge of R, we have $x = 0$, so we consider

$$h(y) = f(0, y) = y^2 - 4y$$

We find $h'(y) = 2y - 4$, so that h has a critical number at $y = 2$. The possible extrema for f along the left edge of R are $(0, 2)$ and the endpoints $(0, 0)$ (we have this one already) and $(0, 4)$. We have

$$\begin{aligned} f(0, 2) &= -4 \\ f(0, 4) &= 0 \end{aligned}$$

Along the hypotenuse of R, we have $y = 4 - x$, so we consider

$$\begin{aligned} k(x) = f(x, 4-x) &= x^2 - 2x + (4-x)^2 - 4(4-x) \\ &= x^2 - 2x + 16 - 8x + x^2 - 16 + 4x \\ &= 2x^2 - 6x \end{aligned}$$

We find $k'(x) = 4x - 6$, so $x = \dfrac{3}{2}$ is a critical number for k. The point $\left(\dfrac{3}{2}, \dfrac{5}{2}\right)$ is a possible extremum for f. We have

$$f\left(\frac{3}{2}, \frac{5}{2}\right) = -\frac{9}{2}$$

In this long list of points, we find the maximum value of f on R is 8, achieved at the point $(4, 0)$, and the minimum value is -5, achieved at the point $(1, 2)$.

5. Use the method of Lagrange multipliers to find the points on the ellipse

$$(x-8)^2 + 9y^2 = 117$$

that are closest to the origin.

Solution: We are asked to minimize distance to the origin, so instead, we'll minimize the square of the distance to the origin. We set $f(x,y) = x^2 + y^2$. We are given $g(x,y) = (x-8)^2 + 9y^2$. We have

$$\begin{aligned} \nabla f(x,y) &= \langle 2x, 2y \rangle \\ \nabla g(x,y) &= \langle 2(x-8), 18y \rangle \end{aligned}$$

so the system of equations we get for the Lagrange method is

$$\begin{aligned} 2x &= 2\lambda(x-8) \\ 2y &= 18\lambda y \\ (x-8)^2 + 9y^2 &= 117 \end{aligned}$$

The second equation says that

$$\begin{aligned} 0 &= 2y - 18\lambda y \\ &= 2y(1-9\lambda) \end{aligned}$$

so either $y = 0$ or $\lambda = \dfrac{1}{9}$. If $y = 0$, then the third equation says $(x-8)^2 = 117$, so $x = 8 \pm \sqrt{117}$.

If $\lambda = \dfrac{1}{9}$, then the first equation says

$$\begin{aligned} 2x &= \frac{2}{9}(x-8) \\ 9x &= x - 8 \\ x &= -1 \end{aligned}$$

Did someone say "rigged"? If $x = -1$, then the third equation says

$$\begin{aligned} (-9)^2 + 9y^2 &= 117 \\ 9y^2 &= 36 \\ y^2 &= 4 \end{aligned}$$

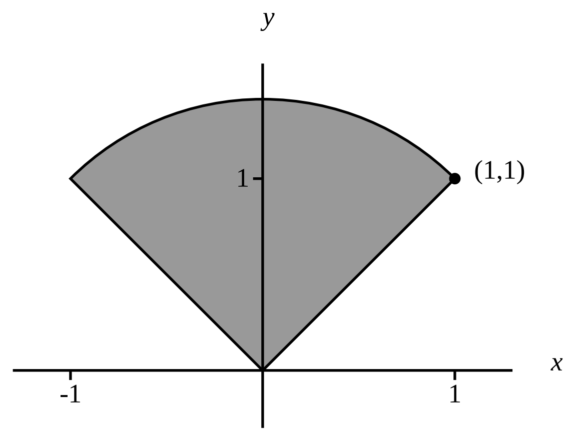
So that y is (conveniently) ± 2.

We have four points: $(8+\sqrt{117}, 0)$, $(8-\sqrt{117}, 0)$, $(-1, 2)$, and $(-1, -2)$. Which of these minimizes the (squared) distance to the origin? Taking $f(x, y) = x^2 + y^2$, we get

$$\begin{aligned} f(8+\sqrt{117}, 0) &= 181 + 16\sqrt{117} \\ &\approx 354 \\ f(8-\sqrt{117}, 0) &= 181 - 16\sqrt{117} \\ &\approx 7.9 \\ f(-1, 2) &= 5 \\ f(-1, -2) &= 5 \end{aligned}$$

The points on the ellipse that are closest to the origin are $(-1, \pm 2)$.

6. Let R be the region in the xy-plane shown at right. Assume the curve is part of a circle centered at the origin. Set up, but do not evaluate, the integral $\displaystyle\iint_R y+1\,dA$ in two ways:



(a) In rectangular coordinates.

Solution: For x between -1 and 0, R is bounded by $y=-x$ on the bottom and $y=\sqrt{2-x^2}$ on the top. For x between 0 and 1, the lower bound is $y=x$. We can write the rectangular integral in two pieces as

$$\iint_R y+1\,dA \quad = \quad \int_{-1}^{0}\int_{-x}^{\sqrt{2-x^2}}(y+1)\,dy\,dx+\int_0^1\int_x^{\sqrt{2-x^2}}(y+1)\,dy\,dx$$

(b) In polar coordinates.

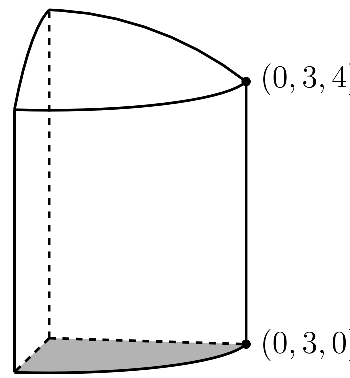
Solution: The region is a polar rectangle with $\dfrac{\pi}{4}\le\theta\le\dfrac{3\pi}{4}$ and $0\le r\le\sqrt{2}$. The integral is

$$\iint_R y+1\,dA \quad = \quad \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\sqrt{2}}(1+r\sin\theta)r\,dr\,d\theta$$

7. Let $E = \{(x, y, z) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2 + y^2 \leq 9 \text{ and } x^2 + y^2 + z^2 \leq 25\}$. Set up (but do not evaluate) $\iiint\limits_E x\, dV$ in three ways:

(a) In rectangular coordinates.

Solution: Here is a sketch of E. The footprint is clearly a quarter disk of radius 3. The top surface is $z = \sqrt{25 - x^2 - y^2}$.
We get



$$\iiint\limits_E x\, dV \quad = \quad \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{25-x^2-y^2}} x\, dz\, dy\, dx$$

(b) In cylindrical coordinates.
Solution: We have

$$\iiint\limits_E x\, dV \quad = \quad \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{\sqrt{25-r^2}} (r\cos\theta) r\, dz\, dr\, d\theta$$

(c) In spherical coordinates.

Solution: Clearly we have $0 \leq \theta \leq \dfrac{\pi}{2}$ and $0 \leq \varphi \leq \dfrac{\pi}{2}$. The lower limit in ρ is always 0, and the upper limit on ρ depends on φ. For values of φ between 0 and the edge of the spherical cap, the upper limit on ρ is 5. For values of φ beyond the edge of the spherical cap, the upper limit on ρ is given by the cylindrical equation $x^2 + y^2 = 9$. In spherical coordinates, this is $\rho^2 \sin^2 \varphi = 9$, or $\rho \sin \varphi = 3$ (since both ρ and $\sin \varphi$ are positive in this region). The upper limit on ρ is thus given by $\rho = 3 \csc \varphi$.

To find the value of φ where the upper limit on ρ changes over, we observe that the edge of the spherical cap of E has radius 3 and lies in the plane $z = 4$. The change-over value of φ is therefore $\tan^{-1}\left(\dfrac{3}{4}\right)$.

We get

$$\begin{aligned}\iiint\limits_E x\,dV &= \int_0^{\frac{\pi}{2}} \int_0^{\tan^{-1}\left(\frac{3}{4}\right)} \int_0^5 (\rho \sin \varphi \cos \theta)\rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\ &+ \int_0^{\frac{\pi}{2}} \int_{\tan^{-1}\left(\frac{3}{4}\right)}^{\frac{\pi}{2}} \int_0^{3 \csc \varphi} (\rho \sin \varphi \cos \theta)\rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta\end{aligned}$$