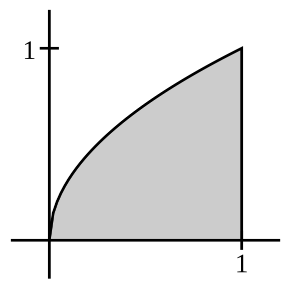
Math 203 **November 2003**

Practice Problems after Hour Exam II – Solutions

1. Compute

$$\int_0^1 \int_{y^2}^1 e^{x^{\frac{3}{2}}}\,dx\,dy.$$

Solution: As it stands, the integral is very nasty. We reverse the order of integration, and hope for the best. The region $0 \le y \le 1$, $y^2 \le x \le 1$ is pictured here.



The same region can be described as $0 \le x \le 1$, $0 \le y \le \sqrt{x}$. Setting up the integral this way, we get

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{x}} e^{x^{\frac{3}{2}}}\,dy\,dx &= \int_0^1 \sqrt{x}e^{x^{\frac{3}{2}}}\,dx \\ &= \left[\frac{2}{3}e^{x^{\frac{3}{2}}}\right]_0^1 \\ &= \frac{2}{3}(e-1).\end{aligned}$$

2. Let C be the helix centered on the z-axis, circling counterclockwise, making two full rotations while rising from $(4,0,0)$ to $(4,0,8)$.

   (a) Parametrize C.
   Solution: We get

$$\mathbf{r}(t) = 4\cos t\vec{\imath} + 4\sin t\vec{\jmath} + \frac{8t}{4\pi}\vec{k},$$

   with $0 \le t \le 4\pi$.

(b) Let $\mathbf{F}(x,y,z) = -y\vec{\imath} + x\vec{\jmath} - z\vec{k}$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Solution: We have

$$\mathbf{r}'(t) = -4\sin t\vec{\imath} + 4\cos t\vec{\jmath} + \frac{2}{\pi}\vec{k}$$

and

$$\mathbf{F}(\mathbf{r}) = -4\sin t\vec{\imath} + 4\cos t\vec{\jmath} - \frac{2t}{\pi}\vec{k}$$

so

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{4\pi} 16 - \frac{4t}{\pi^2}\,dt \\ &= 64\pi - \left[\frac{2t^2}{\pi^2}\right]_0^{4\pi} \\ &= 64\pi - 32\end{aligned}$$

(c) Let $\mathbf{G} = 2xy^2z^2\vec{\imath} + 2yx^2z^2\vec{\jmath} + (2zx^2y^2 + 1)\vec{k}$. Find $\int_C \mathbf{G}\cdot d\mathbf{r}$.

Solution: It looks like $\mathbf{G}$ is a gradient field. In fact, the potential function is $g(x,y,z) = x^2y^2z^2 + z$, so we get

$$\begin{aligned}\int_C \mathbf{G}\cdot d\mathbf{r} &= g(4,0,8) - g(4,0,0) \\ &= 8\end{aligned}$$

3. Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

(a) Write a parametrization for S. Set it up so that the domain is a rectangle in the uv-plane.

Solution: Let u be the circumferential direction and v the radial direction. The surface is made up of circles with $x = v\cos u$ and $y = v\sin u$, where $0 \le u \le 2\pi$ and $1 \le v \le 3$. The z coordinate is given by

$$\begin{aligned}z(u,v) &= 9 - (x(u,v))^2 - (y(u,v))^2 \\ &= 9 - v^2\end{aligned}$$

A parametrization for S is

$$\mathbf{r}(u,v) = (v\cos u)\vec{\imath} + (v\sin u)\vec{\jmath} + (9 - v^2)\vec{k}$$

with $0 \le u \le 2\pi$ and $1 \le v \le 3$.

(b) Set up, but do not evaluate, an integral for the surface area of S.

Solution: From

$$\mathbf{r}(u,v) = v\cos u\vec{\imath} + v\sin u\vec{\jmath} + (9 - v^2)\vec{k}$$

we get

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= (-v\sin u\vec{\imath} + v\cos u\vec{\jmath}) \times (\cos u\vec{\imath} + \sin u\vec{\jmath} - 2v\vec{k}) \\ &= -2v^2\cos u\vec{\imath} - 2v^2\sin u\vec{\jmath} - v\vec{k}\end{aligned}$$

so that

$$\begin{aligned}|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{4v^4 + v^2} \\ &= v\sqrt{4v^2 + 1}\end{aligned}$$

and the area of S is given by

$$\int_0^{2\pi}\int_1^3 v\sqrt{4v^2+1}\,dv\,du$$

4. Let C_1 be the half-circle $x^2 + y^2 = 4$, $y \geq 0$, $z = 0$, oriented counterclockwise from above. Let C_2 be the line segment from $(-2,0,0)$ to $(2,0,0)$. Let C be the union of these two curves, so that C is a closed curve.

Let $\mathbf{F} = (2 - 3y)\vec{\imath} + 2x\vec{\jmath} + xy\vec{k}$.

(a) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

We parametrize C_1 as $\mathbf{r}_1(t) = 2\cos t\vec{\imath} + 2\sin t\vec{\jmath}$ with $0 \leq t \leq \pi$. Thus

$$\mathbf{F}(\mathbf{r}_1) = (2 - 6\sin t)\vec{\imath} + 4\cos t\vec{\jmath} + 4\sin t\cos t\vec{k}$$

and $\mathbf{r}_1{}' = -2\sin t\vec{\imath} + 2\cos t\vec{\jmath}$.

We get

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi} 12\sin^2 t - 4\sin t + 8\cos^2 t\,dt \\ &= \int_0^{\pi} 8 + 4\sin^2 t - 4\sin t\,dt \\ &= 8\pi + 4\frac{\pi}{2} - 4\cdot 2 \\ &= 10\pi - 8\end{aligned}$$

We parametrize C_2 as $\mathbf{r}_2(t) = t\vec{\imath}$ with $-2 \le t \le 2$. Thus

$$\mathbf{F}(\mathbf{r}_2) = 2\vec{\imath} + 2t\vec{\jmath}$$

and $\mathbf{r}_2{}' = \vec{\imath}$.
We get

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{2} 2\,dt = 8$$

Taking the sum, we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 10\pi - 8 + 8 = 10\pi$$

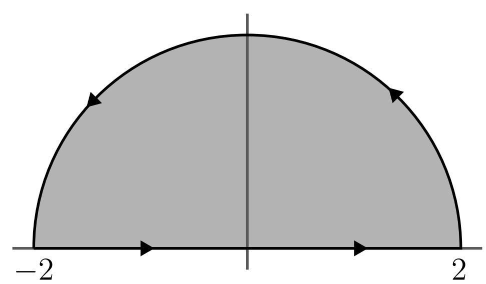
(b) Compute curl $\mathbf{F}$.
Solution: We have

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2-3y & 2x & xy \end{vmatrix} \\ &= x\vec{\imath} - y\vec{\jmath} + 5\vec{k}. \end{aligned}$$

(c) Use Stokes's theorem to verify your answer to part 4a.
Solution: The curve C bounds a half-disk in the xy-plane. We can parametrize this surface as

$$\mathbf{r}(u,v) = (u\cos v)\vec{\imath} + (u\sin v)\vec{\jmath} + 0\vec{k}$$

with $0 \le v \le \pi$ and $0 \le u \le 2$ so that



$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} \\ &= u\vec{k} \end{aligned}$$

We note that this upward orientation agrees with the counterclockwise orientation of C. When we integrate curl $\mathbf{F}$ over the surface S bounded by C, we get

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\pi} \int_0^2 5u\, du\, dv \\ &= 10\pi\end{aligned}$$

There's an even easier way to do this computation: Since the $\vec{k}$ component of curl $\mathbf{F}$ is constant and the surface S lies parallel to (indeed, *in*) the xy-plane, we know that the flux of the field curl $\mathbf{F}$ upward through the surface S is simply the area of S times the $\vec{k}$ component of $\mathbf{F}$. That is,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= 5(\text{Area of } S) \\ &= 5\frac{\pi(2)^2}{2} \\ &= 10\pi\end{aligned}$$

5. Let W be the solid region $x^2 + y^2 \leq 4$ and $0 \leq z \leq 5$. Let

$$\mathbf{F}(x, y, z) = (2x + y\cos(z))\vec{\imath} + (3y + x\sin(z))\vec{\jmath} + z(5 - z)\vec{k}.$$

(a) Explain why the flux of $\mathbf{F}$ through the top and bottom faces of W is zero.

Solution: The area vectors for the top and bottom faces are oriented upward and downward, respectively, so they are multiples of $\vec{k}$. On the top surface, $z = 5$, and so the $\vec{k}$ component of $\mathbf{F}$ is zero. Thus $\mathbf{F} \cdot d\mathbf{S} = 0$ on the top surface, and there is no flux through the top surface. Similarly, on the bottom surface, $z = 0$, so that $\vec{k}$ component of $\mathbf{F}$ is zero, and there is no flux through the bottom surface.

(b) Use the divergence theorem to compute the total flux of $\mathbf{F}$ through the boundary of W.

Solution: We have

$$\begin{aligned}\text{div } \mathbf{F} &= 2 + 3 + 5 - 2z \\ &= 10 - 2z\end{aligned}$$

Thus

$$\iiint_W \text{div } \mathbf{F}\, dV = \int_0^{2\pi} \int_0^2 \int_0^5 (10 - 2z) r\, dz\, dr\, d\theta$$

$$
\begin{aligned}
&= 2\pi \int_0^2 r\left[10z - z^2\right]_0^5 dr \\
&= 2\pi \int_0^2 25r\, dr \\
&= \left[25\pi r^2\right]_0^2 \\
&= 100\pi
\end{aligned}
$$

(c) Determine the flux of $\mathbf{F}$ outward through the lateral surface of the boundary of W (that is, the part of the surface that isn't the top or the bottom).

Solution: The total flux is 100π, and none of it goes out through the top or bottom, so the flux through the lateral surface is 100π.

6. Let S_1 be hemisphere $x^2+y^2+z^2 = 25$, $z \geq 0$, oriented upward (that is, in the positive z direction). Let S_2 be the disk $x^2+y^2 \leq 25$, $z = 0$, oriented downward (that is, in the negative z direction).

Let S be the union of S_1 and S_2, and let W be the region bounded by S.

Let

$$
\mathbf{F} = x^2\vec{\imath} + (2y + z\cos(x))\vec{\jmath} + (y+1)\vec{k}.
$$

(a) Calculate $\displaystyle\iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$.

Solution: We may parametrize S_2 as

$$
\mathbf{r}(u,v) = u\cos v\vec{\imath} + u\sin v\vec{\jmath} + 0\vec{k}
$$

for $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, so that

$$
\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= (\cos v\vec{\imath} + \sin v\vec{\jmath} + 0\vec{k}) \times (-u\sin v\vec{\imath} + u\cos v\vec{\jmath} + 0\vec{k}) \\
&= u\vec{k},
\end{aligned}
$$

which points upward. This is the wrong orientation, so we will take $-u\vec{k}$ as our vector surface element. We get

$$
\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} &= -\int_0^{2\pi}\int_0^5 (u\sin v + 1)u\, du\, dv \\
&= -\int_0^5\int_0^{2\pi} u^2\sin v + u\, du\, dv \\
&= -\int_0^5 2\pi u\, du \\
&= -25\pi
\end{aligned}
$$

(b) Use the divergence theorem to compute $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: We have

$$\text{div } \mathbf{F} = 2x + 2,$$

so by the divergence theorem,

$$\begin{aligned} \iint\limits_S \mathbf{F} \cdot d\mathbf{A} &= \iiint\limits_W 2x + 2 \, dV \\ &= \iiint\limits_W 2x \, dV + \iiint\limits_W 2 \, dV. \end{aligned}$$

The first of these integrals is zero by symmetry; the second is twice the volume of W, which is $\frac{4\pi}{3}(5)^3 = \frac{500\pi}{3}$.

(c) Find $\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

Solution: The flux through S_1 is equal to the flux through S minus the flux through S_2, so we get

$$\begin{aligned} \iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \frac{500\pi}{3} + 25\pi \\ &= \frac{575\pi}{3} \end{aligned}$$