Section 15.5, problem 51: If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

Solution: By the chain rule, we have

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta.\end{aligned}$$

Differentiating again with respect to r, we get

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\sin\theta \\ &= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta \\ &= \left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 z}{\partial x\,\partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\right)\sin\theta \\ &= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta.\end{aligned}$$

Turning to the θ derivatives, we have

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta).\end{aligned}$$

Differentiating again with respect to θ, we get

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)(-r\sin\theta) + \frac{\partial z}{\partial x}(-r\cos\theta) + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)(r\cos\theta) + \frac{\partial z}{\partial y}(-r\sin\theta) \\ &= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial \theta}\right)(-r\sin\theta) + \frac{\partial z}{\partial x}(-r\cos\theta) \\ &\quad + \left(\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta}\right)(r\cos\theta) + \frac{\partial z}{\partial y}(-r\sin\theta) \\ &= \left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}(r\cos\theta)\right)(-r\sin\theta) + \frac{\partial z}{\partial x}(-r\cos\theta) \\ &\quad + \left(\frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta) + \frac{\partial^2 z}{\partial y^2}(r\cos\theta)\right)(r\cos\theta) + \frac{\partial z}{\partial y}(-r\sin\theta) \\ &= \frac{\partial^2 z}{\partial x^2}r^2\sin^2\theta - 2\frac{\partial^2 z}{\partial x\,\partial y}r^2\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}r^2\cos^2\theta - \frac{\partial z}{\partial x}(r\cos\theta) - \frac{\partial z}{\partial y}(r\sin\theta).\end{aligned}$$

Substituting the expressions above into the *right* side of the equation we're trying to prove, we get

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta \\
&\quad + \frac{\partial^2 z}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\cos^2\theta - \frac{\partial z}{\partial x}\frac{\cos\theta}{r} - \frac{\partial z}{\partial y}\frac{\sin\theta}{r} \\
&\quad + \frac{\partial z}{\partial x}\frac{\cos\theta}{r} + \frac{\partial z}{\partial y}\frac{\sin\theta}{r} \\
&= \frac{\partial^2 z}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 z}{\partial y^2}(\cos^2\theta + \sin^2\theta) \\
&= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

as required.